International Journal for Research in Engineering Application & Management (IJREAM)
Special Issue – ICRTET-2018ISSN : 2454-9150

‘‘FLEXURE OF THICK BEAM SUBJECTED TO SINE LOAD

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Abstract - A trigonometric shear deformation theory for flexural analysis of thick beams, taking into account transverse shear deformation effects, is developed. The trigonometric sine function is used in displacement field in terms of thickness coordinate to represent the shear deformation effects. The governing differential equations and boundary conditions are obtained by using the well known principle of virtual work. A general solution technique is developed to solve the governing differential equations of the theory. Theory is applied to thick simply supported beam with sine loading to obtain the complete flexural response subjected to static bending. The behavior of transverse shear stresses at the ends is studied precisely by using equilibrium equation of theory of elasticity and in accordance with the refined shear deformation theories. The results of displacement and stresses are compared with those of elementary theory of beam bending, first order shear deformation theory, third order shear deformation theory and hyperbolic shear deformation theory. Thus, the efficacy of the trigonometric shear deformation theory for the flexure of thick beams is established.

Key Words: Flexure, trigonometric shear deformation, displacement, stress.

I. INTRODUCTION

Elementary theory of bending of beam based on Euler-Bernoulli hypothesis disregards the effects of the shear deformation and stress concentration. The theory is suitable for thin beams and is not suitable for thick or deep beams since it is based on the assumption that the transverse normal to neutral axis remains so during bending and after bending, implying that the transverse shear strain is zero. Since theory neglects the transverse shear deformation, it underestimates deflections in case of thick beams where shear deformation effects are significant.

Bresse [1], Rayleigh [2] and Timoshenko [3] were the pioneer investigators to include refined effects such as rotatory inertia and shear deformation in the beam theory. Timoshenko showed that the effect of transverse vibration of prismatic bars. This theory is now widely referred to as Timoshenko beam theory or first order shear deformation theory (FSDT) in the literature. In this theory transverse shear strain distribution is assumed to be constant through the beam thickness and thus requires shear correction factor to appropriately represent the strain energy of deformation. Cowper [4] has given refined expression for the shear correction factor for different crosssections of beam. The accuracy of Timoshenko beam theory for transverse vibrations of simply supported beam in respect of the fundamental frequency is verified by Cowper [5] with a plane stress exact elasticity solution. To remove the discrepancies in classical and first order shear deformation theories, higher order or refined shear deformation theories were developed and are available in the open literature for static and vibration analysis of beam.

Levinson [6], Bickford [7], Rehfield and Murty [8], Krishna Murty [9], Baluch, Azad and Khidir [10], Bhimaraddi and Chandrashekhara [11] presented parabolic shear deformation theories assuming a higher variation of axial displacement in terms of thickness coordinate. These theories satisfy shear stress free boundary conditions on top and bottom surfaces of beam and thus obviate the need of shear correction factor.

There is another class of refined theories, which includes trigonometric functions to represent the shear deformation effects through the thickness. Vlasov and Leont’ev [12], Stein [13] developed refined shear deformation theories for thick beams including sinusoidal function in terms of thickness coordinate in displacement field. However, with these theories shear stress free boundary conditions are not satisfied at top and bottom surfaces of the beam. A study of literature by Ghugal and Shimpi [14] indicates that the research work dealing with flexural analysis of thick beams using refined trigonometric and hyperbolic shear deformation theories is very scarce and is still in infancy. Ghugal and Dahake [15], Dahake and Ghugal [16] and Jadhav and Dahake [17] employed the trigonometric shear deformation theory for flexure of thick simply supported and cantilever beams. In this paper development of theory and its application to thick simply supported beam is presented.

II. THEOROTICAL FORMULATION

The beam under consideration as shown in Fig. 1 occupies in 0-*x*- *y z* Cartesian coordinate system the region:

$$0 \le x \le L\ ; \quad 0 \le y \le b\ ; \quad -\frac{h}{2} \le z \le \frac{h}{2}$$

Where *x, y, z,* are Cartesian coordinates, *L* and *b* are the length and width of beam in the x and *y* directions respectively, and *h* is the thickness of the beam in the z-direction. The beam is made up of homogeneous, linearly elastic isotropic material.

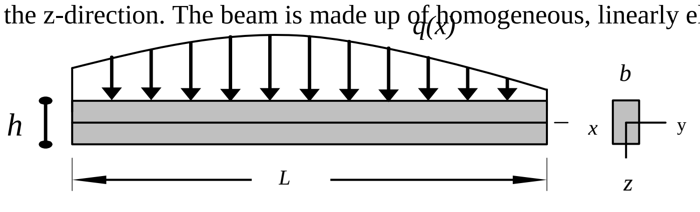

Fig. 1 Beam under bending in *x-z* plane

A. Displacement Field used

The displacement field of the present beam theory is of the form:

$$u(x,z) = -z\frac{dw}{dx} + \frac{h}{\pi}\sin\frac{\pi z}{h}\phi(x)$$
$$w(x,z) = w(x) \tag{1}$$

where u is the axial displacement in x direction and w is the transverse displacement in z direction of the beam. The sinusoidal function is assigned according to the shear stress distribution through the thickness of the beam. The function represents rotation of the beam at neutral axis, which is an unknown function to be determined. The normal and shear strains obtained within the framework of linear theory of elasticity using displacement field given by (1) are as follows.

Normal strain: $$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{d^2w}{dx^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{d\phi}{dx} \tag{2}$$

Shear strain: $$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{dw}{dx} = \cos\frac{\pi z}{h}\phi \tag{3}$$

The stress-strain relationships used are as follows:

$$\sigma_x = E\varepsilon_x = -Ez\frac{d^2w}{dx^2} + \frac{Eh}{\pi}\sin\frac{\pi z}{h}\frac{d\phi}{dx} \tag{4}$$

$$\tau_{zx} = G\gamma_{zx} = G\cos\frac{\pi z}{h}\phi \tag{5}$$

B. Governing Equations

Using (2) through (4) and (5) and using the principle of virtual work, variationally consistent governing differential equations and boundary conditions for the beam under consideration can be obtained. The principle of virtual work when applied to the beam leads to:

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}\left(\sigma_x\delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx}\right)dx\,dz - \int_{x=0}^{x=L}q(x)\,\delta w\,dx = 0$$

Where, the symbol δ denotes the variational operator. Employing Green's theorem to (4) successively‖, we obtain the coupled Euler-Lagrange equations which are the governing differential equations and associated boundary conditions of the beam. The governing differential equations obtained are as follows:

$$EI\frac{d^4w}{dx^4} - \frac{24}{\pi^3}EI\frac{d^3\phi}{dx^3} = q(x) \tag{6}$$

$$\frac{24}{\pi^3}EI\frac{d^3w}{dx^3} - \frac{6}{\pi^2}EI\frac{d^2\phi}{dx^2} + \frac{GA}{2}\phi = 0 \tag{7}$$

The associated consistent natural boundary conditions obtained are of following form:

At the ends $x = 0$ and $x = L$

$$V_x = EI\frac{d^3w}{dx^3} - \frac{24}{\pi^3}EI\frac{d^2\phi}{dx^2} = 0 \quad \text{or } w \text{ is prescribed} \tag{8}$$

$$M_x = EI\frac{d^2w}{dx^2} - \frac{24}{\pi^3}EI\frac{d\phi}{dx} = 0 \quad \text{or } \frac{dw}{dx} \text{ is prescribed} \tag{9}$$

$$M_s = EI\frac{24}{\pi^3}\frac{d^2w}{dx^2} - \frac{6}{\pi^2}EI\frac{d\phi}{dx} = 0 \quad \text{or } \phi \text{ is prescribed} \tag{10}$$

Thus the boundary value problem of the beam bending is given by the above variationally consistent governing differential equations and boundary conditions.

C. The General Solution

The general solution for transverse displacement $w(x)$ and warping function $\phi(x)$ is obtained using (6) and (7) using method of solution of linear differential equations with constant coefficients. Integrating and rearranging (6), we obtain the following expression

$$\frac{d^3w}{dx^3} = \frac{24}{\pi^3}\frac{d^2\phi}{dx^2} + \frac{Q(x)}{EI} \tag{11}$$

where $Q(x)$ is the generalized shear force for beam and it is given by $Q(x)=\int^{x} q\,dx + C_1$ and now (7) is rearranged in the following form.

$$\frac{d^3 w}{dx^3} = \frac{\pi}{4}\frac{d^2\phi}{dx^2} - \beta\phi \tag{12}$$

A single equation in terms of ϕ is now obtained using (11) and (12) as:

$$\frac{d^2\phi}{dx^2} - \lambda^2\phi = \frac{Q(x)}{\alpha EI} \tag{13}$$

the constants α, β and λ appeared in equation (12) and (13) are as follows.

$$\alpha = \left(\frac{\pi}{4} - \frac{24}{\pi^3}\right),\ \beta = \left(\frac{\pi^3}{48}\frac{GA}{EI}\right) \text{ and } \lambda^2 = \frac{\beta}{\alpha}.$$

The general solution of (13) is as follows.

$$\phi(x) = C_2\cosh\lambda x + C_3\sinh\lambda x - \frac{Q(x)}{\beta EI} \tag{14}$$

The equation of transverse displacement $w(x)$ is obtained by substituting the expression of $\phi(x)$ in (12) and then integrating it thrice with respect to x. The general solution for $w(x)$ is obtained as follows.

$$EI\, w(x) = \iiiint q\,dx\,dx\,dx\,dx + \frac{C_1 x^3}{6} + \left(\frac{\pi}{4}\lambda^2 - \beta\right)\frac{EI}{\lambda^3}\left(C_2\sinh\lambda x + C_3\cosh\lambda x\right) + C_4\frac{x^2}{2} + C_5 x + C_6 \tag{15}$$

Where C_1, C_2, C_3, C_4, C_5 and C_6 are arbitrary constants and can be obtained by imposing boundary conditions of beam.

III. EXAMPLE STUDIED

In order to prove the efficacy of the present theory, the following numerical examples are considered. The following material properties for beam are used.

E = 210 GPa, μ = 0.3 and ρ = 7800 kg/m^3, where E is the Young's modulus, ρ is the density, and μ is the Poisson's ratio.

A. Simply supported beam subjected to sine load

The simply supported beam is as shown in Fig. 2 and subjected to sine load, on surface z = -h/2 acting in the downward z direction.

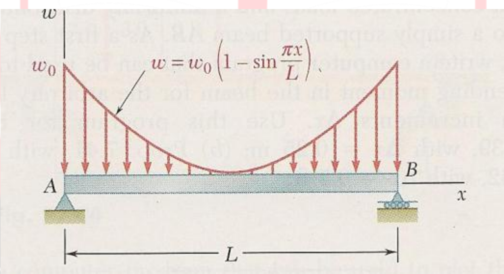

Figure 2: Simply supported beam with sine load

The final expressions for transverse displacement $w(x)$ and $\phi(x)$ are obtained as follows:

$$w(x) = \frac{qL^4}{120EI}\left[\begin{array}{l} 5\frac{x^4}{L^4} - \frac{120}{\pi^4}\sin\frac{\pi x}{L} - 10\frac{x^3}{L^3} + 5\frac{x}{L} - \frac{480}{\pi^5}\frac{E}{G}\frac{h^2}{L^2}\sin\frac{\pi x}{L} \\ - \frac{11520}{\pi^6}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{\sinh\lambda x - \cosh\lambda x + 1}{\lambda^2 L^2} + \frac{1}{2}\frac{x^2}{L^2}\right) \end{array}\right] \tag{16}$$

$$\phi(x) = \frac{qL}{\beta EI}\left(\frac{1}{\pi}\cos\frac{\pi x}{L} + \frac{x}{L} - \frac{1}{2} + \frac{\cosh\lambda x - \sinh\lambda x}{\lambda L}\right) \tag{17}$$

The axial displacement and stresses obtained based on above solutions are as follows:

$$\begin{aligned} u = \frac{1}{10}\frac{z}{h}\frac{qh}{Eb}\frac{L^3}{h^3}&\left[\begin{array}{l} 20\frac{x^3}{L^3} - \frac{120}{\pi^3}\cos\frac{\pi x}{L} - 30\frac{x^2}{L^2} + 5 - \frac{480}{\pi^4}\frac{E}{G}\frac{h^2}{L^2}\cos\frac{\pi x}{L} \\ + \frac{11520}{\pi^6}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{\cosh\lambda x - \sinh\lambda x}{\lambda^2 L^2} + \frac{x}{L}\right) \end{array}\right] \\ &- \frac{48}{\pi^4}\sin\frac{\pi z}{h}\frac{E}{G}\frac{L}{h}\left(\frac{\cosh\lambda x - \sin\lambda x}{\lambda L} + \frac{x}{L} + \frac{1}{\pi}\cos\frac{\pi x}{L} - \frac{1}{2}\right) \end{aligned} \tag{18}$$

Expression for axial stress,

$$\begin{aligned} \sigma_x = \frac{1}{10}\frac{z}{h}\frac{q}{b}\frac{L^2}{h^2}&\left[\begin{array}{l} 60\frac{x^2}{L^2} + \frac{120}{\pi^2}\sin\frac{\pi x}{L} - 60\frac{x}{L} - \frac{480}{\pi^3}\frac{E}{G}\frac{h^2}{L^2}\sin\frac{\pi x}{L} \\ + \frac{11520}{\pi^6}\frac{E}{G}\frac{h^2}{L^2}\left(\sinh\lambda x - \cosh\lambda x + 1\right) \end{array}\right] \\ &- \frac{48}{\pi^4}\sin\frac{\pi z}{h}\frac{E}{G}\frac{L}{h}\left(\sinh\lambda x - \cosh\lambda x + 1 - \sin\frac{\pi x}{L}\right) \end{aligned} \tag{19}$$

Expression for transverse shear stress obtained via constitutive relations

$$\tau_{zx}^{CR} = \frac{48}{\pi^3}\frac{q}{b}\frac{L}{h}\left(\cos\frac{\pi z}{h}\right)\left(\frac{\cosh\lambda x - \sinh\lambda x}{\lambda L} + \frac{x}{L} + \frac{1}{\pi}\pi\cos\frac{\pi x}{L} - \frac{1}{2}\right) \tag{20}$$

Expression for transverse shear stress obtained from equilibrium equation,

$$\tau_{zx}^{EE} = \frac{1}{10}\frac{q}{b}\frac{L}{h}\left(4\frac{z^2}{h^2}-1\right)\left[\begin{array}{l} 120\frac{x}{L} + \frac{120}{\pi}\cos\frac{\pi x}{L} - 60 - \frac{480}{\pi^2}\frac{E}{G}\frac{h^2}{L^2}\cos\frac{\pi x}{L} \\ + \frac{11520}{\pi^6}\frac{E}{G}\frac{h^2}{L^2}(\sinh\lambda x - \cosh\lambda x + 1) \end{array}\right]$$

$$+\frac{48}{\pi^3}\cos\frac{\pi z}{h}\frac{E}{G}\frac{h}{L}\left(\lambda L\cosh\lambda x - \lambda L\sinh\lambda x - \pi\cos\frac{\pi x}{L}\right) \tag{21}$$

IV. RESULTS

In this paper, the results for axial displacement, transverse displacement, inplane and transverse stresses are presented in the following non dimensional form for the purpose of presenting the results in this work.

$$\bar{u} = \frac{Ebu}{q_0 h}, \quad \bar{w} = \frac{10Ebh^3 w}{q_0 L^4}, \quad \bar{\sigma}_x = \frac{b\sigma_x}{q_0}, \quad \bar{\tau}_{zx} = \frac{b\tau_{zx}}{q_0}$$

Table 1: Non-Dimensional Axial Displacement ($\bar{u}$) at (x = 0.25L, z = h/2), Transverse Deflection ($\bar{w}$) at (x = 0.25L, z =0.0) Axial Stress ($\bar{\sigma}_x$) at (x = 0.25L, z = h/2) Maximum Transverse Shear Stress $\bar{\tau}_{zx}^{EE}$ (x = 0, z = 0) of the Simply Supported Beam for Aspect Ratio 4.

Model	L/h	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
TSDT	4	3.1269	0.3611	-1.8815	-1.030	-12.4397
HPSDT		2.1634	0.3218	-2.7606	-1.004	-6.2894
HSDT		1.9743	0.3222	-2.1851	-1.007	-10.5866
FSDT		2.2427	0.3819	-2.1221	-1.365	-3.0000
ETB		2.2427	0.2422	-2.1221	—	-3.0000
TSDT	10	26.7537	0.2613	-13.0225	-2.7171	-9.1344
HPSDT		34.8450	0.2550	-13.9016	-2.6337	-8.0273
HSDT		34.3724	0.2550	-13.3261	-2.6420	-9.2828
FSDT		35.0433	0.2645	-13.2631	-3.4116	-7.5000
ETB		35.0433	0.2422	-13.2631	—	-7.5000

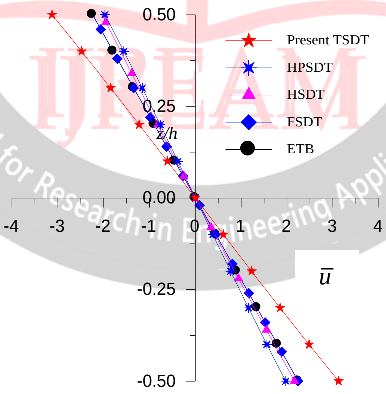

Fig. 3: Variation of axial displacement (u) through the thickness of beam at (x = 0.25L, z) for aspect ratio 4.

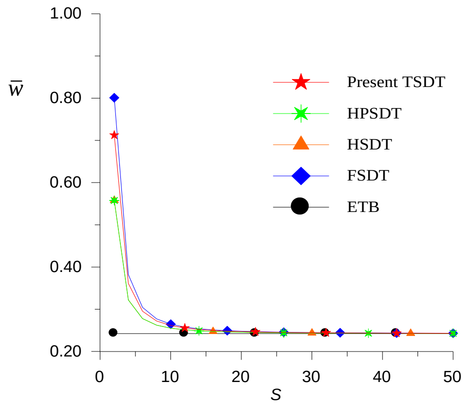

Fig. 4: Variation of transverse displacement (u) through the thickness of beam at (x = 0.25L, z) with respect to aspect ratio.

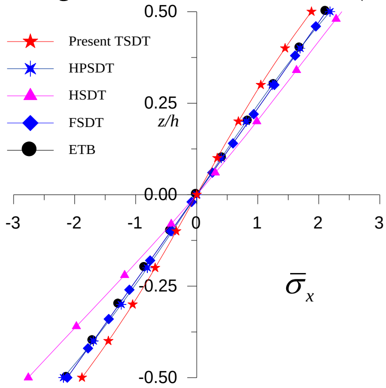

Fig. 5: Variation of axial stress (σ_x) through the thickness at(x = 0.25L, z) for aspect ratio 4.

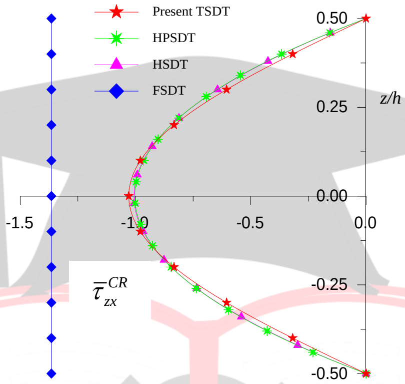

Fig. 6: Variation of transverse shear stress ($\overline{\tau}_{zx}$) through the thickness of beam at (x = 0, z) using CR for aspect ratio 4.

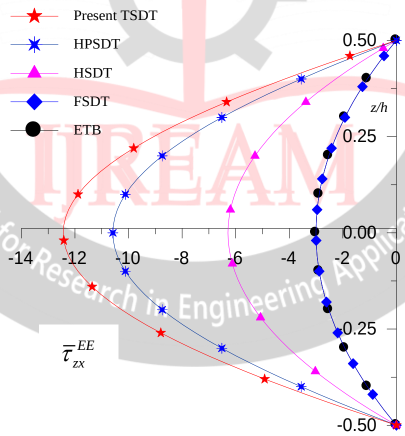

Fig. 7: Variation of transverse shear stress ($\overline{\tau}_{zx}$) through the thickness of beam at (x = 0, z) obtain using EE for aspect ratio 4.

V. DISCUSSION OF RESULTS

The comparison of results of maximum non-dimensional axial displacement ($\overline{u}$) for the aspect ratios 4 and 10 is presented in Table 1for the beam subjected to sine load (see Fig. 2). The values of axial displacement given by present theory are in good agreement with the values of other refined theory for aspect ratio 4 and 10. The through thickness distribution of this displacement obtained by present theory is in close agreement with other refines theories as shown in Fig. 3 for aspect ratio 4.

The comparison of results of maximum non-dimensional transverse displacement ($\overline{w}$) for the aspect ratios 4 and 10 is presented in Table 1. The values of present theory are in excellent agreement with the values of other refined theories for aspect ratio 4 and 10 except those of classical beam theory ETB for aspect ratio 4. The variation of ($\overline{w}$) with aspect ratio S is shown in Fig. 4.

The results of axial stress ($\overline{\sigma}_x$) are shown in Table 1for aspect ratio 4 and 10. The axial stresses given by present theory are compared with other higher order shear deformation theories. It is observed that the results by present theory are in excellent agreement with other refined theories as well as ETB and FSDT. The through thickness variation of this stress given by all the theories at x=0.25L. The variations of this stress are shown in Figure 5. The comparison of maximum non-dimensional transverse shear stress for beam with sine load obtained by the present theory and other refined theories in Table1 for aspect ratios 4 and 10 respectively. The maximum transverse shear stress obtained by present theory using constitutive relation is in good agreement with that of higher order theories for aspect ratio 4 and 10. The through thickness variation of this stress obtained via constitutive relation are presented graphically in Fig. 6 and those obtained via equilibrium equation are presented in Fig. 7. The through thickness variation of this stress when obtained by various theories via equilibrium equation shows the variations with each other. The maximum value of this stress occurs at the neutral axis.

VI. CONCLUSION

The variationally consistent theoretical formulation of the theory with general solution technique of governing differential equations is presented. The general solutions for beam with sine load are obtained in case of thick simply supported beam. The displacements and stresses obtained by present theory are in excellent agreement with those of other equivalent refined and higher order theories. The present theory yields the realistic variation of axial displacement and stresses through the thickness of beam. Thus, the validity of the present theory is established.

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