‘‘BENDING OF THICK ALUMINUM BEAM

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Abstract - Many parts of spacecrafts, airplane is made up of aluminum, which is thick or deep in section. For the analysis of deep or thick beams, a trigonometric shear deformation theory is used, taking into account transverse shear deformation effects, is developed. To represent the shear deformation effects, a sinusoidal function is used in displacement field in terms of thickness coordinate. The important feature of this theory is that the transverse shear stresses can be obtained directly from the use of constitutive relations with excellent accuracy, satisfying the shear stress conditions on the end surfaces of the beam. Hence, the theory obviates the need of shear correction factor. Using the principle of virtual work governing differential equations and boundary conditions are obtained. The thick aluminum beam is considered for the numerical study to show the accuracy of the theory. The cantilever beam subjected to parabolic loads is examined using the present theory. Results obtained are discussed with those of other theories.

Key Words: Bending, trigonometric shear deformation, displacement, stress.

I. INTRODUCTION

Euler-Bernoulli hypothesis disregards the effects of the shear deformation and stress concentration which is in elementary theory of beam bending hence it is suitable for thin beams and is not suitable for deep beams since it is based on the assumption that the transverse normal to neutral axis remains so during bending and after bending, implying that the transverse shear strain is zero. Since theory neglects the transverse shear deformation. It underestimates deflections in case of thick beams where shear deformation effects are significant.

Timoshenko [1] showed that the effect of transverse vibration of prismatic bars. This theory is now widely referred to as Timoshenko beam theory or first order shear deformation theory (FSDT) in the literature. But in this theory transverse shear strain distribution is assumed to be constant through the thickness of beam and thus requires shear correction factor to appropriately represent the strain energy of deformation.

Cowper [2] has given refined expression for the shear correction factor for different cross-sections of beam. The accuracy of Timoshenko beam theory for transverse vibrations of simply supported beam in respect of the fundamental frequency is verified by Cowper [3] with a plane stress exact elasticity solution.

To remove the discrepancies in classical and first order shear deformation theories, higher order or refined shear deformation theories were developed and available in the open literature for static and vibration analysis of beam. Krishna Murthy [4], Baluch *et al*. [5], Bhimaraddi and Chandrashekhara [6] were presented parabolic shear deformation theories assuming a higher variation of axial displacement in terms of thickness coordinate. These theories satisfy shear stress free boundary conditions on top and bottom surfaces of beam and thus obviate the need of shear correction factor.

Kant and Gupta [7], and Heyliger and Reddy [8] presented finite element models based on higher order shear deformation uniform rectangular beams. However, these displacement based finite element models are not free from phenomenon of shear locking [9, 10].

Dahake and Ghugal [11] studied flexural analysis of thick simply supported beam using trigonometric shear deformation theory. Ghugal and Dahake [12, 13] given the flexural solution for the beam subjected to parabolic loading. Sawant and Dahake [14] developed the new hyperbolic shear deformation theory. Chavan and Dahake [15, 16] presented clamped-clamped beam using hyperbolic shear deformation theory. The displacement and stresses for thick beam given by Nimbalkar and Dahake [17].

Jadhav and Dahake [18] presented bending analysis of deep cantilever beam using steel as material. Manal et al [19] investigated the deep fixed beams using new displacement field. Patil and Dahake [20] carried out finite element analysis using 2D plane stress elements for thick beam. Dahake et al [21] studied flexural analysis of thick fixed beam subjected to cosine load. Tupe et al [22] compared various displacement fields for static analysis of thick isotropic beams.

In literature, most of the researchers have used steel as a beam material. As many parts of the spacecrafts, airplane structures are made up of aluminum due to its low weight density. In this research, an attempt has been made to analyze the aluminum deep cantilever beam subjected to parabolic load.

II. Development of Theory

The beam under consideration occupies in $0-x-y-z$ Cartesian coordinate system the region:

$$0 \le x \le L \; ; \quad 0 \le y \le b \; ; \quad -\frac{h}{2} \le z \le \frac{h}{2}$$

where x, y, z are Cartesian coordinates, L and b are the length and width of beam in the x and y directions respectively, and h is the thickness of the beam in the z-direction. The beam is made up of homogeneous, linearly elastic isotropic material.

2.1 The displacement field

The displacement field of the present beam theory is of the form as given below:

$$u(x,z) = -z\frac{dw}{dx} + \frac{h}{\pi}\sin\frac{\pi z}{h}\phi(x)$$
$$w(x,z) = w(x) \tag{1}$$

where u is the axial displacement in x direction and w is the transverse displacement in z direction of the beam. The sinusoidal function is assigned according to the shear stress distribution through the thickness of the beam. The ϕ represents rotation of the beam at neutral axis, which is an unknown function to be determined.

Normal strain

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{d^2w}{dx^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{d\phi}{dx} \tag{2}$$

Shear strain

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{dw}{dx} = \cos\frac{\pi z}{h}\phi \tag{3}$$

Stress-Strain Relationships

$$\sigma_x = E\varepsilon_x$$
$$\tau_{zx} = G\gamma_{zx} \tag{4}$$

2.2 Governing Equations and Boundary Conditions

Using the expressions for strains and stresses (2) through (4) and using the principle of virtual work, variationally consistent governing differential equations and boundary conditions for the beam under consideration can be obtained. The principle of virtual work when applied to the beam leads to:

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}\left(\sigma_x\delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx}\right)dx\,dz - \int_{x=0}^{x=L} q(x)\delta w\,dx = 0 \tag{5}$$

where the symbol δ denotes the variational operator. Employing Green's theorem in Eqn. (4) successively, we obtain the coupled Euler-Lagrange equations which are the governing differential equations and associated boundary conditions of the beam. The governing differential equations obtained are as follows:

$$EI\frac{d^4w}{dx^4} - \frac{24}{\pi^3}EI\frac{d^3\phi}{dx^3} = q(x) \tag{6}$$

$$\frac{24}{\pi^3}EI\frac{d^3w}{dx^3} - \frac{6}{\pi^2}EI\frac{d^2\phi}{dx^2} + \frac{GA}{2}\phi = 0 \tag{7}$$

Cantilever beams:

At Free end: $EI\frac{d^2w}{dx^2} = EI\frac{d\phi}{dx} = EI\frac{d^3w}{dx^3} = EI\frac{d^2\phi}{dx^2} = 0$ at $x = L$ and at Fixed end: $\frac{dw}{dx} = \phi = w = 0$ at $x = 0$ (8)

Thus the boundary value problem of the beam bending is given by the above variationally consistent governing differential equations and boundary conditions.

2.3 The General Solution of Governing Equilibrium Equations of the Beam

The general solution for transverse displacement $w(x)$ and warping function $\phi(x)$ is obtained using Eqns. (6) and (7) using method of solution of linear differential equations with constant coefficients. Integrating and rearranging the first governing Eqn. (6), we obtain the following equation

$$\frac{d^3w}{dx^3} = \frac{24}{\pi^3}\frac{d^2\phi}{dx^2} + \frac{Q(x)}{EI} \tag{9}$$

where $Q(x)$ is the generalized shear force for beam and it is given by $Q(x) = \int_0^x q\,dx + C_1$.

Now second governing Eqn. (7) is rearranged in the following form:

$$\frac{d^3w}{dx^3} = \frac{\pi}{4}\frac{d^2\phi}{dx^2} - \beta\phi \tag{10}$$

A single equation in terms of ϕ is now obtained using Eqns. (11) and (12) as:

$$\frac{d^2\phi}{dx^2} - \lambda^2 \phi = \frac{Q(x)}{\alpha EI} \tag{11}$$

where constants α , β and λ in Eqns. (10) and (11) are as follows

$$\alpha = \left(\frac{\pi}{4} - \frac{24}{\pi^3}\right),\ \beta = \left(\frac{\pi^3}{48}\frac{GA}{EI}\right) \text{and } \lambda^2 = \frac{\beta}{\alpha}$$

The general solution of Eqn. (11) is as follows:

$$\phi(x) = C_2 \cosh \lambda x + C_3 \sinh \lambda x - \frac{Q(x)}{\beta EI} \tag{12}$$

The equation of transverse displacement $w(x)$ is obtained by substituting the expression of $\phi(x)$ in Eqn. (12) and then integrating it thrice with respect to x. The general solution for $w(x)$ is obtained as follows:

III.

$$EI\, w(x) = \iiiint q\, dx\, dx\, dx\, dx + \frac{C_1 x^3}{6} + \left(\frac{\pi}{4}\lambda^2 - \beta\right)\frac{EI}{\lambda^3}\left(C_2 \sinh\lambda x + C_3 \cosh\lambda x\right) + C_4\frac{x^2}{2} + C_5 x + C_6 \tag{13}$$

where, C_1, C_2, C_3, C_4, C_5 and C_6 are arbitrary constants and can be obtained by imposing natural (forced) and / or geometric or kinematical boundary / end conditions of beam.

III. Illustrative Example

In order to prove the efficacy of the present theory, a numerical example is considered. For the static flexural analysis, a uniform beam of rectangular cross section, having span length '*L'*, width '*b'* and thickness '*h'* of homogeneous, elastic and isotropic material is considered. The material properties for beam are as follows:

Table 1: Properties of Aluminum 6061-T6, 6061-T651 [23]

Physical Properties	Value
Density	2700 kg/m^3
Ultimate Tensile Strength	310 MPa
Modulus of Elasticity	68.9 GPa
Notched Tensile Strength	324 MPa
Ultimate Bearing Strength	607 MPa
Poisson's Ratio	0.33
Shear Modulus	26 GPa
Shear Strength	207 MPa

A. Cantilever beam subjected to parabolic load

The beam has its origin at left hand side fixed support at $x = 0$ and free at $x = L$. The beam is subjected to parabolic load, on surface $z = +h/2$ acting in the downward z direction with maximum intensity of load q_0.

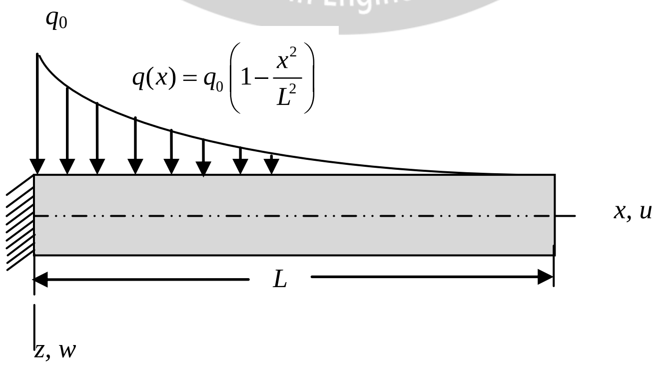

Fig. 1: Cantilever beam with par

General expressions obtained for $w(x)$ and $\phi(x)$ are as follows:

$$w(x) = \frac{q_0 L^4}{24EI}\left\{-\frac{1}{15}\frac{x^6}{L^6} + \frac{x^4}{L^4} - \frac{8}{3}\frac{x^3}{L^3} + 3\frac{x^2}{L^2} + \frac{8}{5}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{x}{L} + \frac{\cosh\lambda x - \sinh\lambda x - 1}{\lambda L}\right)\right\} \tag{14}$$

$$\phi(x) = \frac{q_0 L}{3\beta EI}\left(1 - 3\frac{x}{L} + \frac{x^3}{L^3} + \zeta(x)\right) \tag{15}$$

where $\zeta(x) = (\sinh\lambda x - \cosh\lambda x)$

The axial displacement and stresses obtained based on above solutions are as follows

$$u = \frac{q_0 h}{Eb}\left\{\begin{array}{c} -\frac{z}{h}\frac{L^3}{h^3}\left[-\frac{1}{5}\frac{x^5}{L^5}+2\frac{x^3}{L^3}-4\frac{x^2}{L^2}+3\frac{x}{L}+\frac{1}{5}\frac{E}{G}\frac{h^2}{L^2}\left(1+\zeta(x)\right)\right] \\ +\frac{16}{\pi^4}\frac{E}{G}\frac{L}{h}\sin\frac{\pi z}{\mathrm{h}}\left(1-3\frac{x}{L}+3\frac{x^3}{L^3}+\zeta(x)\right)\end{array}\right\} \tag{16}$$

$$\sigma_x = \frac{q_0}{b}\left\{\begin{array}{c} -\frac{z}{h}\left[\frac{L^2}{h^2}\left(-\frac{x^4}{L^4}+6\frac{x^2}{L^2}-8\frac{x}{L}+3\right)-\frac{4}{5}\frac{E}{G}\left[\lambda L\,\zeta(x)\right]\right] \\ +\frac{16}{\pi^4}\frac{E}{G}\sin\frac{\pi z}{h}\left(-3-3\frac{x^2}{L^2}-\lambda L\zeta(x)\right)\end{array}\right\} \tag{17}$$

$$\tau_{zx}^{CR} = \frac{16}{\pi^3}\frac{q_0}{b}\frac{L}{h}\cos\frac{\pi z}{h}\left(1-3\frac{x}{L}+\frac{x^3}{L^3}+\zeta(x)\right) \tag{18}$$

$$\tau_{zx}^{EE} = 2\frac{q_0 L}{bh}\left\{\begin{array}{c} \left[\frac{z^2}{h^2}-\frac{1}{4}\right]\left[-\frac{x^3}{L^3}+3\frac{x}{L}-2+\frac{1}{5}\frac{E}{G}\frac{h^2}{L^2}\left(\lambda^2L^2\right)\zeta(x)\right] \\ +\frac{8}{\pi^5}\frac{E}{G}\frac{h^2}{L^2}\cos\frac{\pi z}{h}\left(6\frac{x}{L}+\left(\lambda^2L^2\right)\zeta(x)\right)\end{array}\right\} \tag{19}$$

IV. RESULTS

In this paper, the results for axial displacement, transverse displacement, inplane and transverse stresses are presented in the following non dimensional form for the purpose of presenting the results in this work.

$$\bar{u} = \frac{Ebu}{q_0 h},\quad \bar{w} = \frac{10Ebh^3 w}{q_0 L^4},\quad \bar{\sigma}_x = \frac{b\sigma_x}{q_0},\quad \bar{\tau}_{zx} = \frac{b\tau_{zx}}{q_0}$$

Table 2: Non-Dimensional Axial Displacement ($\bar{u}$) at ($x = L$, $z = h/2$), Transverse Deflection ($\bar{w}$) at ($x = L$, $z = 0$) Axial Stress ($\bar{\sigma}_x$) at ($x = 0$, $z = h/2$) Maximum Transverse Shears Stresses $\bar{\tau}_{zx}^{CR}$ ($x=0.01L$, $z=0$) and $\bar{\tau}_{zx}^{EE}$ ($x= 0$, $z=0$) of the Cantilever Beam Subjected to Parabolic Load for Aspect Ratios.

Model	L/h	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
TSDT	4	25.5378	6.7969	35.8715	1.8449	-2.9180
HPSDT		25.5659	6.8083	39.0900	2.2657	-4.7704
HSDT		25.5495	6.8019	36.4960	1.9929	-3.0253
FSDT		25.6000	7.1458	24.0000	0.7286	3.9400
ETB		25.6000	6.3333	24.0000	—	3.9400
TSDT	10	399.8446	6.4094	180.1367	7.9684	3.3635
HPSDT		399.9148	6.4108	188.1392	8.7012	3.9907
HSDT		399.8738	6.4100	181.5160	8.1970	3.9364
FSDT		400.0000	6.4633	150.0000	1.1385	9.8500
ETB		400.0000	6.3333	150.0000	—	9.8500

Graphical variations is only for aspect ratio 4 as follows:

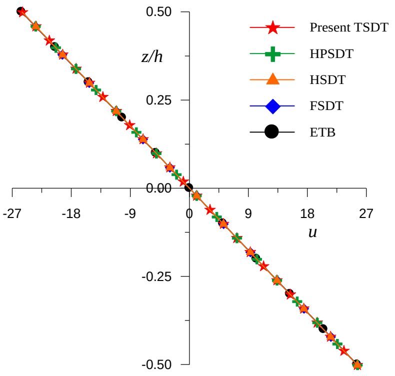

Fig. 2: Variation of axial displacement ($\bar{u}$) through the thickness of cantilever beam at ($x = L$, z)

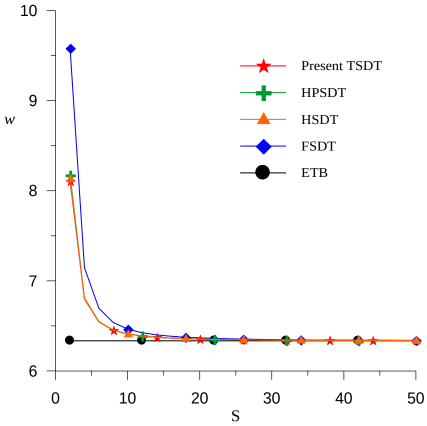

Fig. 3: Variation of maximum transverse displacement ($\bar{w}$) of cantilever beam at ($x = L$, $z = 0$) with aspect ratio S.

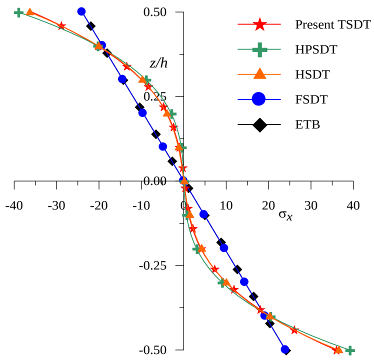

Fig. 4: Variation of axial stress ($\bar{\sigma}_x$) through the thickness of cantilever beam at ($x = 0$, z)

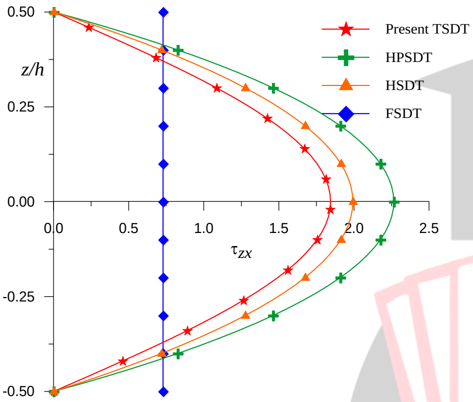

Fig. 5: Variation of transverse shear stress ($\bar{\tau}_{zx}$) through the thickness of cantilever beam at ($x = 0.01L$, z) obtain using constitutive relation.

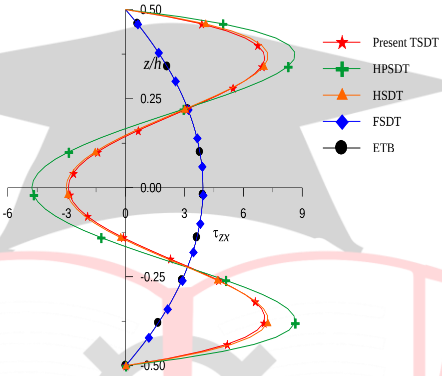

Fig. 6: Variation of transverse shear stress ($\bar{\tau}_{zx}$) through the thickness of cantilever beam at ($x = 0$, z) obtain using equilibrium equation.

V. DISCUSSION OF RESULTS

The comparison of results of maximum non-dimensional axial displacement ($\bar{u}$) for the aspect ratios of 4 and 10 is presented in Table 2 parabolic load (see Fig. 1). Among the results of all the other theories, the values of axial displacement, at the free end of the beam, given by present theory are in close agreement with the values of ETB, FSDT and other refined theories for aspect ratio 4 and 10.The through thickness distribution of this displacement obtained by present theory is in close agreement with classical and other refined theories as shown in Fig. 2 for aspect ratio 4. The comparison of results of maximum non-dimensional transverse displacement ($\bar{w}$) for the aspect ratios of 4 and 10 is presented in Table 1. Among the results of all the other theories, the values of present theory are in excellent agreement with the values of other refined theories for aspect ratio 4 and 10 except those of classical beam theory (ETB) and FSDT of Timoshenko. The variation of $\bar{w}$ with aspect ratio (S) is shown in Fig. 3. For the aspect ratios greater than 20 all the refined theories converges to the values of classical beam theory. The results of axial stress ($\bar{\sigma}_x$) are shown in Table 2. The axial stresses given by present theory are compared with other higher order shear deformation theories. Present and other higher order refined theories provide the non-linear variations of axial stress across the thickness at the built-in end due to heavy stress concentration. The comparison of maximum non-dimensional transverse shear stress for a cantilever beam with parabolic load obtained by the present theory and other refined theories is presented in Table 1 for aspect ratio of 4 and 10. The maximum transverse shear stress obtained by present theory using constitutive relation is in close agreement with that of other higher order theory (HSDT). The values obtained by HPSDT using equilibrium equation show considerable departure from the values of present and HSDT. The values of present theory and those of HSDT are in good agreement with each other. The through thickness variation of this stress obtained via constitutive relation are presented graphically in Fig. 5 and those obtained via equilibrium equation are presented in Fig. 6. It can be seen from these figures that the nature of variation obtained by both the approaches is different from each other. The through thickness variation of this stress via equilibrium equation shows the considerable departure, with change in sign, compared to the one given by ETB and FSDT due to heavy stress concentration associated with the built-in end of the beam . The maximum negative value of this stress occurs at the neutral axis. However, ETB and FSDT yield the identical positive values this stress at $z = 0$ and the identical variations across the thickness of the beam. It is seen that the anomalous behavior in the vicinity of built-in end cannot be captured by constitutive relation. Further, lower order theories, ETB and FSDT, cannot predict this behavior even with the use of equilibrium equation.

VI. CONCLUSION

The use of present theory gives accurate results as seen from the numerical examples studied and it is capable of predicting the local effects in the vicinity of the built-in end of the cantilever beam. This validates the efficacy and credibility of trigonometric shear deformation theory.

REFERENCES

1. Timoshenko S. P., "On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars," Philosophical Magazine, Series 6, Vol. 41, 1921, pp. 742-746.
2. Cowper G. R., "The Shear Coefficients in Timoshenko Beam Theory," ASME Journal of Applied Mechanics, Vol. 33, 1966, pp. 335-340.
3. Cowper G. R., "On The Accuracy of Timoshenko's Beam Theory," ASCE journal of the Engineering Mechanics Division, Vol. 94, No. EM 6, 1968, pp. 1447-1453.
4. Murty A. V. K., 1970, "Vibration of Short Beams," AIAA Journal, Vol. 8, 1970, pp. 3438.
5. Baluch M. H., Azad A. K. and Khidir, M. A., "Technical Theory of Beams with Normal Strain," Journal of the Engineering Mechanics, Proceedings of ASCE, Vol. 110, 1984, pp. 1233-1237.
6. Bhimaraddi A. and Chandrashekhara K., "Observations on Higher-Order Beam Theory," Journal of Aerospace Engineering, Proceedings of ASCE, Technical Note., Vol. 6, 1993, pp. 408-413.
7. Kant T. and Gupta A., "A Finite Element Model for a Higher order Shear Deformable Beam Theory," Journal of Sound and Vibration, Vol. 125, No.2, 1988, pp. 193-202.
8. Heyliger P. R., and Reddy J. N., "A Higher Order Beam Finite Element for Bending and Vibration Problems," Journal of Sound and Vibration, Vol. 126, No. 2, 1988, pp. 309-326.
9. Ghugal Y. M., "A Simple Higher Order Theory for Beam with Transverse Shear and Transverse Normal Effect," Departmental Report, No. 4, Applied of Mechanics Department, Government College of Engineering, Aurangabad, India, 2006, pp. 1-96.
10. Averill R.C. and Reddy J. N., "An assessment of four-noded plate finite elements based on a generalized third order theory", International Journal of Numerical Methods in Engineering, 33, 1992, pp. 1553-1572.
11. Dahake A. G. and Ghugal Y. M., "Flexure of Thick Simply Supported Beam Using Trigonometric Shear Deformation Theory", International Journal of Scientific and Research Publications, 2(11), ISSN 2250-3153, 2012, pp. 1-7
12. Ghugal Y. M. and Dahake A. G., "Flexural Analysis of Deep Beam Subjected to Parabolic Load Using Refined Shear Deformation Theory", Applied and Computational Mechanics, 6(2), 2012, pp. 163-172
13. Ghugal Y. M. and Dahake A. G., "Flexure of thick beams using refined shear deformation theory", International Journal of Civil and Structural Engineering, 3(2), 2012, pp. 321-335
14. Sawant M. K. and Dahake A. G., "A New hyperbolic Shear Deformation Theory for Analysis of thick Beam", International Journal of Innovative research in Science, engineering and technology, ISSN: 2319-8753, 3(2), 2014, pp. 9636-9643
15. Chavan V. B. and Dahake A. G., "Analysis of Thick Beam Bending Problem by Using a New Hyperbolic Shear Deformation Theory", International Journal of Engineering Research and General Science, 2(5), 2014, ISSN 2091-2730, pp. 209-215
16. Chavan V. B. and Dahake A. G., " A Refined Shear Deformation Theory for Flexure of Thick Beam", International Journal of Pure and Applied Research in Engineering and Technology, Impact Factor: 4.226, ISSN: 2319-507X, 3(9), 2015, pp. 109-119.
17. Nimbalkar V. N. and Dahake A. G., "Displacement and Stresses for Thick Beam using New Hyperbolic Shear Deformation Theory", International Journal of Pure and Applied Research in Engineering and Technology, Impact Factor: 4.226, ISSN: 2319-507X, 3(9), May 2015, pp. 120-130
18. Jadhav V. A. and Dahake A. G., "Bending Analysis of Deep Beam Using Refined Shear Deformation Theory", International Journal of Engineering Research, ISSN: e2319-6890, p2347-5013, 5(3), doi:10.17950/ijer/v513/003, Feb. 2016, pp. 526-531
19. Manal S. S., Sawant R. M. and Dahake A. G., "A New Trigonometric Shear Deformation Theory for Thick Fixed Beam", International Journal of Engineering Research, ISSN: e2319-6890, p2347-5013, 5(3), doi:10.17950/ijer/v513/004, Feb. 2016, pp. 532-536
20. Patil P. B. and Dahake A. G., "Finite Element Analysis Using 2D Plane Stress Elements for Thick Beam", Journal of Aerospace Engineering and Technology, ISSN: 2231-038X (Online), ISSN: 2348-7887 (Print),STM, 6(2), 2016, pp. 1-8
21. Dahake A. G., Manal S. S. and Sawant R. M., "Flexure of Fixed Thick Beam using Trigonometric Shear Deformation Theory", Proceedings of 6th International Congress on Computational Mechanics and Simulation, Indian Institute of Technology, (IIT) Powai (Bombay), Maharashtra, India, 27th June to 1st July 2016, pp. 1112-1115

22. Tupe D. H., Dahake A. G. and Gandhe G. R., "Comparison of various displacement fields for static analysis of thick isotropic beams", Structural Engineering Convention (SEC-2016), CSIR-SERC, Chennai, INDIA, 21-23 December 2016
23. Properties of Aluminum 6061-T6, 6061-T651, http://www.aerospacemetals.com

