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Application of Algebra for Genetic Inheritance

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Abstract: Medicine was once largely free of Mathematics. Genetics is riddled with Probability and Statics, Matrices and Linear Algebra. Here an attempt is made to explain how Linear Algebra can be applied to Genetics. It is the study of inheritance or the transmission of traits from one generation to next. we discuss simple Medelian inheritance, Non associativity of inheritance, gametic, Zygote algebras with genetic relations and Baric algebra.

Keywords- Genetics, Linear Algebra, Inheritance,Ttraits, Gametic, Zygote, Baric algebra

I. INTRODUCTION

Medicine was once largely free of Mathematics, but chemical trials call for Mathematical Statistics, the circulation of blood is becoming a topic for hydrodynamist, genetics is riddled with Probability and Statistics, Matrices, Linear Algebra and advanced Algebra, Orthopedics and Oncology deals with directional data analysis and many to mention. The study of applications of mathematical modeling and mathematical techniques leads to get an insight into the problems of biosciences. Mathematical biosciences are mainly concerned with Mathematical Modeling in Biology and medicine and deal with those areas of biosciences which have already been mathematicized.

One of the disciplines included in Mathematical Biosciences is Mathematical Genetics which deals with the transfer of genetic characteristics from generation to generation through the action of genes.

Motivated by the applications of Mathematics in various angles of medicine, contributions of mathematical modeling in Genetics are surveyed to initiate research. As Biomathematics is today a growing and essential adjunct to the further development of the biological and medical sciences, here works on studies of mathematical genetics are reviewed and some of the problems are exercised.

II. SURVEY OF LITERATURE

Bernstein worked on Demonstration math ematique de la loid'heredit de Mendel. in 1923.Etheringtonpresented Genetic algebras in 1939 and Non-associative algebra and the symbolism of genetics in 1941. Schafer published Structure of genetic algebras in 1949. Mendel authored Experiments in Plant-Hybridization in1959.Special train algebras arising in genetics was developed by Gonshorin 1960. Holgate presented Sequences of powers in genetic algebras in1967. Genetic algebras associated with sex linkagewas published in 1970. In 1971Lyubichworked on Basic concepts and theorems of the evolutionary genetics of free populations. McHale and Ringwood. Haldane studied linearisation of baric algebras in 1983. Contributions to genetic algebras was published in 1971. W¨orz-Busekroscontributed The zygotic algebra for sex-linkage in 1974. The zygotic algebra for sex-linkage. II was published in 1975. W¨orz-Busekrosdeveloped Algebras in Genetics in1980. Abraham worked on Linearizing quadratic transformations in genetic algebras in 1980. Peresipresented On baric algebras with prescribed automorphisms in 1986.

In 2015 Frederik Nijhout et.al., using mathematical models to understand metabolism, genes, and disease, *BMC Biology*. Nestor et.al., developed The (Mathematical) Modeling Process in Biosciences, in 2015. Traykov et.al., published Mathematical models in genetics, Russian Journal of Genetics 2016

III. SIMPLE MENDELIAN INHERITANCE

As a natural first example, we consider simple Mendelian inheritance for a single gene with two alleles A and a. In this case, two gametes fusing (or reproducing) to form a zygote gives the multiplication table shown in Table 4.1, which in freshman biology class might be called a Punnett square for simple Mendelian inheritance.

Table.1. Alleles passing from gametes to zygotes

	A	A
A	AA	Aa
A	aA	Aa

Table.2.Multiplication table of the gametic algebra

	A	A
A	A	$\frac{1}{2}(A+a)$
A	$\frac{1}{2}(a+A)$	A

The zygotes AA and aa are called **homozygous**, since they carry two copies of the same allele. In this case, simple Mendelian inheritance means that there is no chance

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involved as to what genetic information will be inherited in the next generation; i.e., AA will pass on the allele A and aa will pass on a. However, the zygotes Aa and aA (which are equivalent) each carry two different alleles. These zygotes are called heterozygous. The rules of simple Mendelian inheritance indicate that the next generation will inherit either A or a with equal frequency. So, when two gametes reproduce, a multiplication is induced which indicates how the hereditary information will be passed down to the next generation? This multiplication is given by the following rules:

(1) $A \times A = A$

(2) $A \times a = \frac{1}{2}A + \frac{1}{2}a$

(3) $a \times A = \frac{1}{2}a + \frac{1}{2}A$

(4) $a \times a = a$

Rules (1) and (4) are expressions of the fact that if both gametes carry the same allele, then the O spring will inherit it. Rules (2) and (3) indicate that when gametes carrying A and a reproduce, half of the time the O spring will inherit A and the other half of the time it will inherit a. These rules are an algebraic representation of the rules of simple Mendelian inheritance. This multiplication table is shown in Table.2. We should point out that we are only concerning ourselves with genotypes (gene composition) and not phenotypes (gene expression). Hence we have made no mention of the dominant or recessive properties of our alleles.

Now that we've defined a multiplication on the symbols A and a we can mathematically de ne the two dimensional algebra over $\mathbb{R}$ with basis $\{A, a\}$ and multiplication table as in Table.2. This algebra is called the gametic algebra for simple Mendelian inheritance with two alleles.

But gametic multiplication is just the beginning! In order for actual diploid cells (or organisms) to reproduce, they must first go through a reduction division process.

Table.3. Multiplication table of the zygotic algebra for simple Mendelian inheritance

	AA	Aa	Aa
AA	AA	$\frac{1}{2}(AA+Aa)$	Aa
Aa	$\frac{1}{2}(AA+Aa)$	$\frac{1}{2}AA+\frac{1}{2}Aa+\frac{1}{2}aa$	$\frac{1}{2}(Aa+aa)$
Aa	Aa	$\frac{1}{2}(Aa+aa)$	Aa

So that only one set of alleles is passed on. For humans this occurs when males produce sperm and females produce eggs. When reproduction occurs, the hereditary information is then passed on via the gametic multiplication we've already defined. Therefore, when two zygotes reproduce another multiplication operation is formed taking into consideration both the reduction division process and gametic multiplication. In our example of simple Mendelian inheritance for one gene with the two alleles A and a, zygotes have three possible genotypes: AA, aa, and Aa. Let's consider the case of two zygotes both with genotype Aa reproducing. The reduction division process splits the zygote and passes on one allele for reproduction. In the case of simple Mendelian inheritance the assumption is that both alleles will be passed on with equal frequency. Thus, half the time A gets passed on and half the time a does. We represent this with the "frequency distribution" $\frac{1}{2}A+\frac{1}{2}a$.

Therefore, symbolically $Aa \times Aa$ becomes $\left(\frac{1}{2}A+\frac{1}{2}a\right)\times\left(\frac{1}{2}A+\frac{1}{2}a\right)$

Formally multiplying these two expression together results in $\frac{1}{4}Aa+\frac{1}{2}Aa+\frac{1}{4}aa$

using the notion that $aA = Aa$. In this way, zygotic reproduction produces the multiplication table shown in Table 4.3. So we can de ne the three dimensional algebra over $\mathbb{R}$ with basis $\{AA, Aa, aa\}$ and multiplication table as in Table 4.3. It is called the zygotic algebra for simple Mendelian inheritance with two alleles. The process of constructing a zygotic algebra from the original gametic algebra is called commutative duplication of algebras. We will discuss this process from a mathematical perspective later.

Now that we've seen how the gametic and zygotic algebras are formed in the most basic example, we shall begin to consider the mathematical (and indeed, algebraic) structure of such algebras.

IV. THE NON-ASSOCIATIVITY OF INHERITANCE

Depending on the "population" you are concerned with, a general element $\alpha A + \beta a$ of the gametic algebra which satisfies $\alpha, \beta \in \mathbb{R}$ with $0 \le \alpha, \beta \le 1$ and $\alpha + \beta = 1$ can represent a population, a single individual of a population, or a single gamete. In each case, the coefficients α and β signify the percentage of frequency of the associated allele. i.e., if the element represents a population, then α is the percentage of the population which carries the allele A on the gene under consideration. β Likewise, is the percentage of the population which has the allele a.

For those elements of the gametic and zygotic algebras which represent populations, multiplication of two such elements represents random mating between the two populations. It seems logical that the order in which populations mate is significant. i.e., if population P mates with population Q and then the resulting population mates with R, the resulting population is not the same as the population resulting from P mating with the population obtained from mating Q and R originally. Symbolically, $(P \times Q) \times R$ is not equal to $P \times (Q \times R)$. So, we see that from a purely biological perspective, we should expect that the

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algebras which arise in genetics will not satisfy the associative property.

Now, if we study the multiplication tables of both the gametic and zygotic algebras for simple Mendelian inheritance, we will notice immediately that the algebras are commutative. From a biological perspective, if populations P and Q are mating, it makes no difference whether you say P mates with Q or Q mates with P. How-ever, as we should expect, these algebras do not satisfy the associative property. E.g., in the gametic algebra apply the rules of multiplication and the distributive property to see that $A(A\times a)=\frac{3}{2}A+\frac{1}{4}a$ However, $(A\times A)\times a = A\times a = \frac{1}{2}A+\frac{1}{4}a$. Hence, the associative property does not hold for the gametic algebra. The same is true for the zygotic algebra. In general, the algebras which arise in genetics are commutative but non-associative.

V. Gametic and Zygotic Algebras

In many genetic situations, Mendelian inheritance does not hold. E.g., gene mutation or recombination both result in different inheritance rules. The gametic and zygotic algebras we discussed in the previous section corresponded to the very specific example of simple Mendelian inheritance for a single gene with two alleles. The more general definitions for gametic and zygotic algebras are given. Suppose now we have a random mating population with n distinct gametes. Call them $a_1,\ldots,a_n$. These gametes could differ at one or more genetic loci. Then consider these n gametes as basic elements of an n-dimensional real vector space. Multiplication is defined by

$$a_i a_j = \sum_{k=1}^{n} \gamma_{ijk} a_k$$

Such that

1. $0 \le \gamma_{ijk} \le 1 \quad i,j,k = 1,\ldots,n$
2. $\sum_{k=1}^{n} \gamma_{ijk} a_k = 1$
3. $\gamma_{ijk} = \gamma_{jik}$

The resulting algebra **G** is called an n-dimensional gametic algebra.

For the zygotic algebra we consider pairs of the n gametes, $a_{ij} = a_i a_j$ with the understanding that $a_{ij} = a_{ji}$, so without loss we only consider a_{ij} with $i \le j$. Then random mating of zygotes a_{ij} and a_{pq} will yield zygotes a_{ks} with a certain probability; call it $\gamma_{ij;pq;ks}$. This de nes zygotic multiplication,

$$a_{ij} a_{pq} = \sum_{k\le s}^{n} \gamma_{ij,pq,ks} a_{ks}$$

such that

1. $0 \le \gamma_{ij,pq,ks} \le 1$
2. $\sum_{k,s=1}^{n} \gamma_{ij,pq,ks} = 1$
3. $\gamma_{ij,pq,ks} = \gamma_{pq,ij,ks}$

where in each case $i \le j, p \le q$ and $k \le s$. The resulting algebra $\mathfrak{Z}$ is the zygotic algebra. We note that the zygotic algebra can be constructed from the gametic algebra through a process called commutative duplication, which was originally introduced by Etherington in the general setting of a (not necessarily commutative nor associative) linear algebra. Using this process, one can calculate the zygotic multiplication constants from the gametic multiplication constants in the following way:

$$\gamma_{ij,pq,ks} = \begin{cases} \gamma_{ijk}\gamma_{pqs} + \gamma_{ijs}\gamma_{pqk}, & \text{for k<s} \\ \gamma_{ijk}\gamma_{pqs} & \text{fork=s} \end{cases}$$

In modern terms, commutative duplication can be realized using tensor products. For any commutative algebra A, tensor it with itself (in the sense of vector spaces) to form $A\otimes A$. Then, commutative duplication can be achieved via the quotient $(A\otimes A)/I$, where I is the subspace generated by elements of the form $x\otimes y - y\otimes x$. This quotient space is, in fact, a commutative algebra, where multiplication is defined by $(a,b)(c,d)=(ab,cd)$. Gonshor [1960] first gave this as a basis-free definition of commutative duplication of an algebra.

In addition, beginning with a zygotic algebra $\mathfrak{Z}$, commutative duplication produces another algebra **C** with genetic relevance, which is generally referred to as the copular algebra. The genetic significance of this algebra is that its elements, which are unordered pairs of zygotes, represent the mating types of a population.

VI. Algebra with Genetic Relation

Mathematically, the algebras that arise in genetics (via gametic, zygotic, or copular algebras) are very interesting structures. They are generally commutative but non-associative, yet they are not necessarily Lie, Jordan, or alternative algebras. In addition, many of the algebraic properties of these structures have genetic significance. Indeed, it is the interplay between the purely mathematical structure and the corresponding genetic properties that makes this subject so fascinating. The work is turned now from the motivating genetics to the more formal mathematical study of the underlying algebraic structure.

The most general definition of an algebra which could have genetic significance is that of an algebra with genetic realization. An algebra with genetic realization is an algebra A over the real numbers $\mathbb{R}$ which has a basis $a_1,\ldots,a_n$ and a multiplication table

$$a_i a_j = \sum_{k=1}^{n} \gamma_{ijk} a_k$$

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such that $0 \le \gamma_{ij,pq,ks} \le 1$ for all *i,j,k* and

$$\sum_{k=1}^{n} \gamma_{ijk} = 1$$

for i; j = 1,. . ., n. Such a basis is called the natural basis for A.

It is easy to see that our earlier examples of gametic and zygotic algebras for simple Mendelian inheritance, as well as the general gametic and zygotic algebras, are all algebras with genetic realization. In a general algebra A with genetic realization, an element x in A represents a population, or a gene pool for a population, if its expression as a linear combination of the basic elements $a_1, \ldots, a_n$,

$$x = \xi_1 a_1 + \xi_2 a_2 + \ldots + \xi_n a_n$$

satisfies $0 \le \xi_i \le 1$ for all $i = 1, 2, \ldots, n$ and $\sum_{k=1}^{n} \xi_i = 1$. Then

ξ_i is percentage of the population x which carries the allele a_i

The class of all algebras with genetic realization is too large to say much about. However, since all gametic algebras (and their commutative duplicates) satisfy the definition, it provides a solid framework for what constitutes an algebra with genetic significance.

VII. Baric Algebras

For strictly mathematical purposes, it is unnecessary to restrict the field our algebras are defined over to be the real numbers. Hence, we will work over an arbitrary field k when appropriate. As we have seen, algebras with genetic realization are not necessarily associative algebras. However, they do belong to a rather special class of non-associative algebras. A general non-associative algebra need not possess a matrix representation. Yet, algebras with genetic realization do. In fact, they possess the simplest possible matrix representation {a scalar representation.

An algebra A over a field k is called a baric algebra if it admits a non-trivial algebra homomorphism $w : A \to k$. In other words, a baric algebra is an algebra with a one-dimensional representation. The homomorphism w is called the weight function (or baric function).

Proposition Let A be an n-dimensional algebra with genetic realization over $\mathbb{R}$. Then A is a baric algebra.

Proof. Let $\{a_1, \ldots, a_n\}$ denote a natural basis for A. Define $w : A \to \mathbb{R}$ by $w(a_i) = 1$ for $i = 1, 2, \ldots, n$ and then extend linearly onto A. i.e., $x = \sum_{i=1}^{n} \xi_i a_i$ then $w(x) = \sum_{k=1}^{n} \xi_i w(a_i)$.

Hence $w(x) = \sum_{k=1}^{n} \xi_i$. Then we need only show that w is a hommorphism.

Let $x = \sum_{i=1}^{n} \xi_i a_i$ and $y = \sum_{j=1}^{n} \beta_j a_j$.

Then

$$xy = \sum_{i=1}^{n} \xi_i a_i \sum_{j=1}^{n} \beta_j a_j$$
$$= \sum_{i=1}^{n} \xi_i \left(\sum_{j=1}^{n} \beta_j a_i a_j \right)$$
$$= \sum_{i=1}^{n} \xi_i \left(\sum_{j=1}^{n} \beta_j \left(\sum_{k=1}^{n} \gamma_{ijk} a_k \right) \right)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \xi_i \beta_j \gamma_{ijk} a_k$$

T Then apply w to get that

$$w(xy) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \xi_i \right) \left(\sum_{j=1}^{n} \beta_j \right) \gamma_{ijk}$$
$$= \left(\sum_{i=1}^{n} \xi_i \right) \left(\sum_{j=1}^{n} \beta_j \right) \sum_{k=1}^{n} \gamma_{ijk}$$
$$= \left(\sum_{i=1}^{n} \xi_i' \right) \left(\sum_{j=1}^{n} \beta_j \right)$$

since $\sum_{k=1}^{n} \xi_i = 1$. But, then $w(xy) = w(x) w(y)$. Therefore, w is a homomorphism and is a baric algebra.

From a strictly mathematical perspective, an interesting question to ask about baric algebras is whether or not their weight functions are uniquely determined. The following example shows that in general, they are not.

Example. Let $A = \langle a_1, a_2, a_3 \rangle \mathbb{R}$ be a commutative 3-dimensional algebra with the multiplication table below.

	a_1	a_2	a_3
a_1	$a_1 + a_2$	a_2	a_2
a_2	a_2	a_2	a_2
a_3	a_2	a_2	$a_2 + a_3$

Then define $\omega_1 : A \to \mathbb{R}$ via $\omega_1(a_1) = 1$ and $\omega_1(a_2) = \omega_1(a_3) = 0$. And define $\omega_2 : A \to \mathbb{R}$ via $\omega_2(a_3) = 1$, while $\omega_2(a_1) = \omega_2(a_2) = 0$. It is easy to see that $\omega_1 \ne \omega_2$, and it is a simple verification that they both define homomorphisms.

Even though the above example shows that not all baric algebras have a unique weight function, many of them do. In order to exhibit at least a sufficient condition for a baric algebra to have a unique weight function, the issue of powers in a non-associative algebra are to be discussed.

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