International Journal for Research in Engineering Application & Management (IJREAM)
ISSN : 2454-9150

2(1) - Prime Partial Ideals Of Partial Γ- Semirings

[1]Dr. M. Siva Mala, [2]Dr. P. V. Srinivasa Rao

[1]Assistant Professor, Department of Mathematics, V.R. Siddhartha Engineering College, Kanuru, Vijayawada, Krishna (District), Andhra Pradesh, INDIA. sivamala_aug9@yahoo.co.in

[2]Associate Professor, Basic Engineering Department, DVR & Dr. HS MIC College of Technology, Kanchikacherla, Krishna (District), Andhra Pradesh, INDIA. srinu_fu2004@yahoo.co.in

Abstract: A partial Γ-semiring is a structure possessing an infinitary partial addition and a ternary multiplication, satisfying a set of identities. The partial functions under disjoint-domain sums and functional composition is a partial Γ-semiring. In this paper we introduce the notions of 2(1)-prime partial ideal and m2(m1)-system and obtained some relations between them in partial Γ-semirings.

Keywords: prime ideal, m-system, 2(1)-prime partial ideal, m2(m1)-system of (R, Γ), 2(1)-spec(R) & 2(1)-prime radical of K.

I. INTRODUCTION

Arbib, Manes, Benson[3], [5] and Streenstrup[14] introduced the notions of sum ordered partial monoids & sum ordered partial semirings. M. Murali Krishna Rao[6] in 1995 introduced the notion of a Γ-semiring as a generalization of semirings and Γ-rings.

In [8] and [9] we introduced the notions of partial Γ-semiring and Γ-so-ring as a generalization of partial semirings and Γ-semirings. In [10], [11], [12] and [13] we studied the ideal theory for Γ-so-rings.

In this paper we introduce the notions of 2(1)-prime partial ideal and $m_2(m_1)$-system in a partial Γ-semiring and obtain the characterizations of intersection of all 2(1)-prime partial ideals of R and 2(1)-prime radical of K.

II. PRELIMINARIES

In this section we collect some important definitions and results for our use in this paper.

1.1. Definition. [5] A *partial monoid* is a pair (M, Σ) where M is a nonempty set and Σ is a partial addition defined on some, but not necessarily all families $(x_i : i \in I)$ in M subject to the following axioms:
(i) **Unary Sum Axiom.** If $(x_i : i \in I)$ is a one element family in M and $I = \{ j \}$, then $\Sigma(x_i : i \in I)$ is defined and equals x_j.
(ii) **Partition-Associativity Axiom.** If $(x_i : i \in I)$ is a family in M and $(I_j : j \in J)$ is a partition of I, then $(x_i : i \in I)$ is summable if and only if $(x_i : i \in I_j)$ is summable for every j in J, $(\Sigma(x_i : i \in I_j) : j \in J)$ is summable, and $\Sigma(x_i : i \in I) = \Sigma(\Sigma(x_i : i \in I_j) : j \in J)$.

1.2. Definition. [8] Let (R, Σ) and (Γ, Σ') be two partial monoids. Then R is said to be a *partial Γ-semiring* if there exists a mapping $R \times \Gamma \times R \to R$ (images to be denoted by $x\gamma y$ for $x, y \in R$ and $\gamma \in \Gamma$) satisfying the following axioms:
(i) $x\gamma(y\mu z) = (x\gamma y)\mu z$,
(ii) a family $(x_i : i \in I)$ is summable in R implies $(x\gamma x_i : i \in I)$ is summable in R and $x\gamma[\Sigma(x_i : i \in I)] = \Sigma(x\gamma x_i : i \in I)$,
(iii) a family $(x_i : i \in I)$ is summable in R implies $(x_i\gamma x : i \in I)$ is summable in R and $[\Sigma(x_i : i \in I)]\gamma x = \Sigma(x_i\gamma x : i \in I)$,
(iv) a family $(\gamma_i : i \in I)$ is summable in Γ implies $(x\gamma_i y : i \in I)$ is summable in R and $x[\Sigma'(\gamma_i : i \in I)]y = \Sigma(x\gamma_i y : i \in I)$ for all $x, y, z, (x_i : i \in I)$ in R and $\gamma, \mu, (\gamma_i : i \in I)$ in Γ.

1.3. Definition. [10] Let R be a partial Γ-semiring, A be a nonempty subset of R and Ω be a nonempty subset of Γ. Then the pair (A, Ω) of (R, Γ) is said to be a *left (right) partial Γ-ideal* of R if it satisfies the following:
(i) $(x_i : i \in I)$ is a summable family in R and $x_i \in A \ \forall i \in I$ implies $\Sigma_i x_i \in A$,
(ii) $(\alpha_i : i \in I)$ is a summable family in Γ and $\alpha_i \in \Omega \ \forall i \in I$ implies $\Sigma'_i \alpha_i \in \Omega$, and
(iii) for all $x \in R; y \in A$ and $\alpha \in \Omega$, $x\alpha y \in A$ $(y\alpha x \in A)$.

If (A, Ω) is both left and right partial Γ-ideal of a partial Γ-semiring R, then (A, Ω) is called a *partial Γ-ideal* of R. If $\Omega = \Gamma$, then A is called a *partial ideal* of (R, Γ).

1.4. Definition. [9] A Γ-so-ring R is said to be a *complete partial Γ-semiring* if every family of elements in R is summable and every family of elements in Γ is summable.

1.5. Theorem. [10] Let R be a complete partial Γ-semiring. If A and Ω are subsets of R and Γ respectively, then the partial Γ-ideal generated by (A, Ω) is the pair $(\{ x \in R \mid x = \Sigma_i x_i + \Sigma_j r_j\alpha_j x_j' + \Sigma_k x_k''\alpha_k' r_k' + \Sigma_l r_l''\alpha_l'' x_l'''\alpha_l''' r_l'''$, where $x_i, x_j', x_k'', x_l''' \in A$, $r_j, r_k', r_l'', r_l''' \in R$ and $\alpha_j, \alpha_k', \alpha_l'', \alpha_l''' \in \Gamma \}, \{ \beta \in \Gamma \mid \beta = \Sigma_i'\beta_i, \beta_i \in \Gamma \})$.

1.6. Remark. [10] Let R be a complete partial Γ-semiring and $a \in R$. Then the left/right/both sided ideals of R generated by a are
(i) $< a] = \{ x \in R \mid x = \Sigma_n a + \Sigma_j r_j\alpha_j a, r_j \in R, \alpha_j \in \Gamma, n \in N \}$,
(ii) $[a > = \{ x \in R \mid x = \Sigma_n a + \Sigma_k a\alpha_k' r_k', r_k' \in R, \alpha_k' \in \Gamma, n \in N \}$,

(iii) $< a > = \{ x \in R \mid x = \Sigma_n a + \Sigma_j r_j\alpha_j a + \Sigma_k a\alpha_k' r_k' + \Sigma_l r_l''\alpha_l'' a\alpha_l''' r_l''',\ r_j, r_k', r_l'', r_l''' \in R$ and $\alpha_j, \alpha_k', \alpha_l'', \alpha_l''' \in \Gamma,\ n \in N \}$.

We call $< a >$ as the *principal ideal generated by a*.

1.7. Definition. [10] Let R be a partial Γ-semiring. If A, B are subsets of R and Γ_1 is a subset of Γ, define $A\Gamma_1 B$ as the set $\{x \in R \mid \exists\ a_i \in A, \gamma_i \in \Gamma_1, b_i \in B, \Sigma_i a_i\gamma_i b_i$ *exists and* $x = \Sigma_i a_i\gamma_i b_i \}$.

If $A = \{a\}$ then we also denote $A\,\Gamma_1 B$ by $a\,\Gamma_1 B$. If $B = \{b\}$ then we also denote $A\,\Gamma_1 B$ by $A\,\Gamma_1 b$. Similarly if $A = \{a\}$ and $B = \{b\}$, we denote $A\,\Gamma_1 B$ by $a\,\Gamma_1 b$ and thus
$a\,\Gamma_1 b = \{x \in R \mid x = a\gamma b$ *for some* $\gamma \in \Gamma_1 \}$.

An ideal A of a Γ-so-ring R is called proper if $A \neq R$.

1.8. Definition. [12] A proper partial ideal P of a partial Γ-semiring R is said to be *prime* if and only if for any partial ideals A, B of R, $A\Gamma B \subseteq P$ implies $A \subseteq P$ *or* $B \subseteq P$.

2. 2(1)-PRIME PARTIAL IDEALS

We introduce the notion of 2(1)-prime partial ideals in partial Γ-semirings as follows:

2.1. Definition. Let R be a partial Γ - semiring and A be a proper partial ideal of (R,Γ). Then A is said to be *2-prime* if and only if for any subtractive partial ideals I, J of (R,Γ), $I\Gamma J \subseteq A$ implies $I \subseteq A$ or $J \subseteq A$.

2.2. Definition. Let R be a partial Γ - semiring and A be a proper partial ideal of (R,Γ). Then A is said to be *2-prime* if and only if for any subtractive partial ideal I and a partial ideal J of (R,Γ), $I\Gamma J \subseteq A$ implies $I \subseteq A$ or $J \subseteq A$.

2.3. Remark. Let R be a partial Γ - semiring and A be a proper partial ideal of (R,Γ). If A is a prime ideal of (R,Γ) then A is a 2(1)-prime partial ideal of (R,Γ).

Proof. Suppose A is a prime partial ideal of (R,Γ). Let I, J be any two subtractive partial ideals of (R,Γ) such that $I\Gamma J \subseteq A$. Then I, J are partial ideals of (R,Γ) such that $I\Gamma J \subseteq A$. Since A is prime, $I \subseteq A$ or $J \subseteq A$. Hence A is a 2(1)-prime partial ideal of (R,Γ).

The following example illustrates that in a partial Γ-semiring R, a 2(1)-prime partial ideal of (R,Γ) is not prime.

2.4. Example. Let $R = \{0, 1, 2, 3\}$. Define Σ on R as

$$\sum_{i\in I} x_i = \begin{cases} x_j, & \text{if } x_i = 0 \quad \forall i \neq j \text{ for some } j \\ 2, & \text{if } x_j = x_k = 1 \text{ for some } j,k \text{ and } x_i = 0\ \forall i \neq j,k \\ 3, & \text{if } J = \{i \mid x_i \neq 0\} \text{ is finite and } \Sigma(x_i : i \in J) \geq 3 \\ \text{undefined}, & \text{otherwise} \end{cases}$$

Then (R,Σ) is a partial monoid. Let $\Gamma := \mathrm{N} \bigcup \{0\}$, the set of all non-negative integers. Then Γ is a partial monoid with finite support addition. Define a mapping $R\times\Gamma\times R$ into R as

$$(x,\alpha,y) = \begin{cases} 0 & \text{if any one of } x,\alpha,y \text{ is } 0, \\ x & \text{if } \alpha = y = 1, \\ \alpha & \text{if } x = y = 1, \ \& \ \alpha \leq 3, \\ y & \text{if } x = \alpha = 1, \\ 3 & \text{if } x.\alpha.y \geq 3 \end{cases}$$

Then R is a partial Γ-semiring. The partial ideals of (R,Γ) are {0}, {0, 3}, {0, 2, 3}, R and the subtractive partial ideals of (R,Γ) are {0} and R. Since $\{0,2,3\}.\Gamma.\{0,2,3\} = \{0,3\} = A$ and $\{0,2,3\} \not\subset I$, I is not a prime partial ideal of (R,Γ). But it easy to prove that I is a 2(1)-prime partial ideal of (R,Γ).

2.5. Lemma. Let R be a complete partial Γ-semiring and A be a partial ideal of (R,Γ). Then the subtractive closure of A is $\overline{A} = \{a \in R \mid a + x \in A,$ for some $x \in A\}$.

Proof. Take $S := \{a \in R \mid a + x \in A,$ for some $x \in A\}$. First we prove that S is the subtractive partial ideal of (R,Γ) containing A: Let $(a_i : i \in I)$ be a (summable) family in R such that $a_i \in S\ \ \forall\ i \in I$. Then $\forall\ i \in I, a_i + x_i \in A$ for some $x_i \in A$. Now $\sum_i a_i + \sum_i x_i = \sum_i (a_i + x_i) \in A$ and hence $\sum_i a_i \in S$. Let $r \in R, \alpha \in \Gamma, a \in S$. Then $r \in R, \alpha \in \Gamma$ and $a + x \in A$ for some $x \in A$. Now $r\alpha(a + x) = r\alpha a + r\alpha x \in A$ and $(a + x)\alpha r = a\alpha r + x\alpha r \in A$ and hence $r\alpha a, a\alpha r \in S$. Thus S is a partial ideal of (R,Γ). Let $a, b \in R$ be such that $a + b \in S$ and $b \in S$. Then $a + b + x \in A$ and $b + y \in A$ for some $x, y \in A \Rightarrow a + b + x + y \in A$ for some $b + x + y \in A$ and hence $a \in S$. Thus S is a subtractive partial ideal of (R,Γ). Since $a + 0 = a \in A, A \subseteq S$. Now we prove that S is the smallest subtractive partial ideal of (R,Γ) containing A: Let P be another subtractive partial ideal of (R,Γ) containing A. Let $a \in S$. Then $a + x \in A$ for some

$x \in A$. Since $A \subseteq P, a+x \in P$ and $x \in P$. Since P is subtractive, $a \in P$ and hence $S \subseteq P$. Hence the lemma.

2.6. Theorem. Let A be a partial ideal of a partial Γ-semiring R. Then A is 2(1)-prime partial ideal if and only if $x, y \in R$ such that $\overline{<x>}\Gamma\overline{<y>} \subseteq A (<x>\Gamma<y> \subseteq A)$ then $x \in A$ or $y \in A$.

Proof. Suppose A is a 2(1)-prime partial ideal of (R,Γ). Let $x, y \in R$ such that $\overline{<x>}\Gamma\overline{<y>} \subseteq A$. Since $\overline{<x>}, \overline{<y>}$ are subtractive partial ideals of (R,Γ) such that $\overline{<x>}\Gamma\overline{<y>} \subseteq A$ and A is 2(1)-prime, $\overline{<x>} \subseteq A$ or $\overline{<y>} \subseteq A$. Hence $x \in A$ or $y \in A$.

Conversely, suppose that $x, y \in R$ such that $\overline{<x>}\Gamma\overline{<y>} \subseteq A$ then $x \in A$ or $y \in A$. Let X, Y be any two subtractive partial ideals of (R,Γ) such that $X\Gamma Y \subseteq A$. Suppose $X \not\subset A$. Then $\exists\ x \in X \ni x \notin A$. Let $y \in Y$. Now we prove that $\overline{<x>}\Gamma\overline{<y>} \subseteq X\Gamma Y$: Let $a \in \overline{<x>}\Gamma\overline{<y>}$. Then $a = \sum_i x_i \alpha_i y_i$ for some $x_i \in \overline{<x>}, \alpha_i \in \Gamma, y_i \in \overline{<y>}, \forall i \in I$. Since $x_i \in \overline{<x>}, x_i + x_i' \in <x> \subseteq X$ for some $x_i' \in <x> \subseteq X \quad \forall i \in I$. Since X is subtractive, $x_i \in X$ $\forall i \in I$. Similarly we prove that $y_i \in Y \quad \forall i \in I$. Thus $a = \sum_i x_i \alpha_i y_i \in X\Gamma Y$. Hence $\overline{<x>}\Gamma\overline{<y>} \subseteq X\Gamma Y. \Rightarrow \overline{<x>}\Gamma\overline{<y>} \subseteq A$. By assumption, $x \in X$ or $y \in Y$. Since $x \notin X, y \in Y$ and hence $Y \subseteq A$. Hence the theorem.

2.7. Definition. If X, Y are non-empty subsets of a partial Γ-semiring R, then

(i) $(X:Y)_l = \{a \in R \mid a\Gamma Y \subseteq X\}$

(ii) $(X:Y)_r = \{a \in R \mid Y\Gamma a \subseteq X\}$

(iii) $(X:Y) = \{a \in R \mid a\Gamma Y \subseteq A$ and $Y\Gamma a \subseteq X\}$.

2.8. Lemma. Let R be a partial Γ-semiring. Then

(i) If X and Y are left(right) partial ideals of (R,Γ) then $(X:Y)_l ((X:Y)_r)$ is a partial ideal of (R,Γ).

(ii) If X is a subtractive left(right) partial ideal and Y is a left(right) partial ideal of (R,Γ) then $(X:Y)_l ((X:Y)_r)$ is a subtractive partial ideal of (R,Γ).

Proof. (i): we have $(X:Y)_l = \{a \in R \mid a\Gamma Y \subseteq X\}$. Let $(a_i : i \in I)$ be a summable family in R and $a_i \in (X:Y)_l$ $\forall i \in I$. Then $x_i \Gamma B \subseteq A \quad \forall i \in I. \Rightarrow \sum_{i\in I} a_i \Gamma Y \subseteq X. \Rightarrow (\sum_{i\in I} a_i)\Gamma Y \subseteq X. \Rightarrow \sum_{i\in I} x_i \in (X:Y)_l$. Let $r \in R, \alpha \in \Gamma, a \in (X:Y)_l$. Then $r \in R, \alpha \in \Gamma$ and $a\Gamma Y \subseteq X$. Consider $(r\alpha a)\Gamma Y = r\alpha(a\Gamma Y)$. Since $a\Gamma Y \subseteq X, r\alpha(a\Gamma Y) \subseteq r\alpha X$. Since X is a left partial ideal, $r\alpha X \subseteq X. \Rightarrow (r\alpha a)\Gamma Y \subseteq X. \Rightarrow r\alpha a \in (X:Y)_l$. Consider $(a\alpha r)\Gamma Y = a\alpha(r\Gamma Y)$. Since Y is a left partial ideal of $(R,\Gamma), r\Gamma Y \subseteq Y. \Rightarrow a\alpha(r\Gamma Y) \subseteq a\alpha Y \subseteq a\Gamma Y \subseteq X$. $\Rightarrow (a\alpha r)\Gamma Y \subseteq X. \Rightarrow a\alpha r \in (X:Y)_l$. Therefore $(X:Y)_l$ is a subtractive partial ideal of (R,Γ).

(ii): Suppose X is a subtractive left partial ideal and Y is a left partial ideal of (R,Γ). Let $a, a+b \in (X:Y)_l. \Rightarrow a\Gamma Y \subseteq X$ and $(a+b)\Gamma Y \subseteq X$. Let $c \in b\Gamma Y$. Then $c = \sum_i b\alpha_i y_i$ for some $\alpha_i \in \Gamma, y_i \in Y$. Now $a\alpha_i y_i \in a\Gamma Y \quad \forall i \in I$. $\Rightarrow x\alpha_i b_i \in A \quad \forall i \in A. \Rightarrow \sum_i a\alpha_i y_i \in X$. Now $\sum_i a\alpha_i y_i + \sum_i b\alpha_i y_i = \sum_i (a+b)\alpha_i y_i \in (a+b)\Gamma Y \subseteq X$. Since X is subtractive, $\sum_i a\alpha_i y_i \in X$ and $\sum_i a\alpha_i y_i + c \in X$, we have $c \in X$. Therefore $b\Gamma Y \subseteq X. \Rightarrow b \in (X:Y)_l$. Hence $(X:Y)_l$ is a subtractive partial ideal of (R,Γ).

2.9. Theorem. Let A be a subtractive partial ideal of a partial Γ-semiring R. Then A is prime if and only if A is 2(1)-prime.

Proof. By Remark 2.3., if A is prime then A is 2(1)-prime.

Conversely, suppose that A is 2(1)-prime partial ideal of (R,Γ). Let X, Y be any two partial ideals of (R,Γ) $\ni X\Gamma Y \subseteq A$. Then $X \subseteq (A:Y)_l$. Since A is subtractive, by lemma 2.8.(ii), $(A:Y)_l$ is a subtractive partial ideal containing X. $\Rightarrow \overline{X} \subseteq (A:Y)_l. \Rightarrow \overline{X}\Gamma Y \subseteq A \Rightarrow Y \subseteq (A:\overline{X})_r$.

Since A is subtractive, by lemma 2.8.(ii), $(A:\overline{X})_r$ is a subtractive partial ideal containing Y. $\Rightarrow \overline{Y} \subseteq (A:\overline{X})_r. \Rightarrow \overline{X}\Gamma\overline{Y} \subseteq A$. Since A is 2-prime and $\overline{X}, \overline{Y}$ are subtractive partial ideals $\overline{X} \subseteq A$ or $\overline{Y} \subseteq A$. Thus $X \subseteq A$ or $Y \subseteq A$. Hence A is a prime partial ideal of (R,Γ).

3. $m_2(m_1)$ – SYSTEM OF (R,Γ).

We define $m_2(m_1)$-system in a partial Γ-semiring as follows:

3.1. Definition. Let R be a partial Γ-semiring and A be a subset of R. Then A is called an $m_2(m_1)$-*system* of (R,Γ) if for any $a,b \in A, \exists$ $a_1 \in \overline{<a>}, b_1 \in \overline{<b>}$ $(b_1 \in <b>), r \in R, \alpha,\beta \in \Gamma$ such that $a_1\alpha r\beta b_1 \in A.$

3.2. Remark. Let R be a partial Γ-semiring. Then every m-system of (R,Γ) is an $m_2(m_1)$-system of (R,Γ).

Proof. Let A be an m-system of (R,Γ). Let $a,b \in A.$ Then $\exists$ $\alpha,\beta \in \Gamma, r \in R \ni a\alpha r\beta b \in A. \Rightarrow$ $a_1 = a \in \overline{<a>}, b_1 = b \in \overline{<b>} \ni a_1\alpha r\beta b_1 \in A.$ Hence A is an $m_2(m_1)$-system of (R,Γ).

The following example illustrates that an $m_2(m_1)$-system is not m-system in a partial Γ-semiring R.

3.3. Example. Consider the partial Γ-semiring R as in the example 2.4. Then a subset $A = \{1,2\}$ of R is an $m_2(m_1)$-system. For $2 \in A,$ for any $r \in R$ and $\alpha,\beta \in \Gamma,$ $2\alpha r\beta 2 = 3 \notin A.$ Hence $A = \{1,2\}$ is not an m-system of (R,Γ).

3.4. Lemma. Let R be a partial Γ-semiring with left/right unity and P be a proper partial ideal of (R,Γ). Then P is 2(1)-prime partial ideal of (R,Γ) if and only if $R \setminus P$ is an $m_2(m_1)$-system of R.

Proof. Suppose P is a 2-prime partial ideal of (R,Γ). Let $a,b \in R \setminus P.$ Then $a,b \notin P.$ By theorem 2.6., $\overline{<a>}\Gamma\overline{<b>} \not\subset P. \Rightarrow \exists$ $x \in \overline{<a>}\Gamma\overline{<b>} \ni x \notin P. \Rightarrow x = \sum_i a_i\alpha_i b_i$ for some $a_i \in \overline{<a>}, \alpha_i \in \Gamma, b_i \in \overline{<b>}.$ Since $x \notin P, \sum_i a_i\alpha_i b_i \notin P. \Rightarrow \exists$ $a_i' \in \overline{<a>}, \alpha_i' \in \Gamma, b_i' \in \overline{<b>} \ni a_i'\alpha_i' b_i' \notin P. \Rightarrow a_i'\alpha_i' b_i' \notin P.$ Since R has left unity, $\exists$ a family of elements $(e_j : j \in J)$ of R, $(\gamma_j : j \in J)$ of Γ such that $\sum_j e_j\gamma_j b_i' = b_i'. \Rightarrow a_i'\alpha_i'(\sum_j e_j\gamma_j b_i') \notin P. \Rightarrow$ $\sum_j a_i'\alpha_i' e_j\gamma_j b_i' \notin P.$ $\Rightarrow \exists$ $a_i' \in \overline{<a>}, b_i' \in \overline{<b>}, \alpha_i', \gamma_j \in \Gamma, e_j \in R \ni a_i'\alpha_i' e_j\gamma_j b_i' \in R \setminus P.$ Hence $R \setminus P$ is an m_2-system of R.

Conversely suppose that $R \setminus P$ is an m_2-system of R. Let $a,b \in R \ni \overline{<a>}\Gamma\overline{<b>} \subseteq P.$ Suppose $a \notin P$ and $b \notin P.$ Then $a,b \in R \setminus P. \Rightarrow \exists$ $a_1 \in \overline{<a>}, b_1 \in \overline{<b>}, \alpha,\beta \in \Gamma, r \in R \ni a_1\alpha r\beta b_1 \in R \setminus P.$ Since $a_1 \in \overline{<a>}, a_1\alpha r \in \overline{<a>}.$ $\Rightarrow (a_1\alpha r)\beta b_1 \in \overline{<a>}\Gamma\overline{<b>} \ni a_1\alpha r\beta b_1 \notin P,$ a contradiction. Hence $a \in P$ or $b \in P.$ Therefore P is a 2-prime partial ideal of (R,Γ).

3.5. Theorem. Let R be a partial Γ-semiring with left/right unity, Q be a partial ideal of (R,Γ), and M be an $m_2(m_1)$-system of R $\ni \overline{Q} \cap M = \phi.$ Then there exists a 2(1)-prime partial ideal A of (R,Γ) such that $Q \subseteq A$ and $\overline{A} \cap M = \phi$ and A is maximal with respect to this property.

Proof. Take $\tau = \{P \mid P$ is a partial ideal of (R,Γ) such that $Q \subseteq P$ and $\overline{P} \cap M = \phi\}.$ Clearly $Q \in \tau.$ Hence $(\tau, \subseteq)$ is a non-empty poset. Let $\{P_j \mid j \in J\}$ be a simply ordered family in $\tau.$ Take $P' := \bigcup_{j \in J} P_j.$ Then P' is clearly a partial ideal containing Q. Now we prove that $\overline{P'} \cap M = \phi$: Suppose that $x \in \overline{P'} \cap M.$ Then $x \in \overline{P'}$ and $x \in M.$ $\Rightarrow x + y \in P'$ for some $y \in P'.$ $\Rightarrow x + y \in P_j$ and $y \in P_k$ for some $j,k \in J.$ Since $\{P_j \mid j \in J\}$ is a simply ordered family, either $P_j \subseteq P_k$ or $P_k \subseteq P_j.$ So, assume that $P_j \subseteq P_k.$ $\Rightarrow x + y \in P_k$ and $y \in P_K$ for some $k \in J. \Rightarrow x \in \overline{P_k}$ and thus $x \in \overline{P_k} \cap M = \phi,$ a contradiction. Hence $\overline{P'} \cap M = \phi.$ Therefore every simply ordered family in τ has an upper bound in $\tau.$ By Zorn's lemma, τ has a maximal element. Let it be A.

Now let X, Y be subtractive partial ideals such that $X\Gamma Y \subseteq A$ and $X \not\subset A. \Rightarrow X\Gamma A \subseteq A$ and $A\Gamma X \subseteq A. \Rightarrow X\Gamma A \subseteq \overline{A}$ and $A\Gamma X \subseteq \overline{A}. \Rightarrow A \subseteq (\overline{A} : X)_r$ and $A \subseteq (\overline{A} : X)_l.$ Since $\overline{A}$ is subtractive, $(\overline{A} : X)_r$ and $(\overline{A} : X)_l$ are subtractive partial ideals containing A. Since A is maximal, either $A = (\overline{A} : X)_r$ or $\overline{(\overline{A} : X)_r} \cap M = (\overline{A} : X)_r \cap M \neq \phi$ and either $A = (\overline{A} : X)_l$ or $\overline{(\overline{A} : X)_l} \cap M = (\overline{A} : X)_l \cap M \neq \phi.$

ISSN : 2454-9150

Suppose $(\overline{A}:X)_r \cap M \neq \phi. \Rightarrow \exists$ $x \in (\overline{A}:X)_r \cap M. \Rightarrow x \in (\overline{A}:X)_r$ and $x \in M. \Rightarrow \overline{<x>} \subseteq (\overline{A}:X)_r. \Rightarrow X\Gamma\overline{<x>} \subseteq \overline{A}.$

Case(i): If $X \cap M \neq \phi.$ Then $\exists$ $a \in X \cap M. \Rightarrow a, x \in M$ & M is an m_2-system. $\Rightarrow \exists$ $a_1 \in \overline{<a>}, x_1 \in \overline{<x>}, \alpha, \beta \in \Gamma, r \in R \ni a_1\alpha r\beta x_1 \in M.$ Since $X\Gamma\overline{<x>} \subseteq \overline{A}, (a_1\alpha r)\beta x_1 \in \overline{A}. \Rightarrow \overline{A} \cap M \neq \phi,$ a contradiction. Hence $A = (\overline{A}:X)_r$ & $A = (\overline{A}:X)_l$.

Case(ii): If $X \cap M = \phi.$ Since $X\Gamma\overline{<x>} \subseteq \overline{A}, X \subseteq (\overline{A}:\overline{<x>})_l.$ Since $\overline{<x>} \cap M \neq \phi,$ by case(i), $A = (\overline{A}:\overline{<x>})_l. \Rightarrow X \subseteq A,$ a contradiction. Hence $A = (\overline{A}:X)_r$ & $A = (\overline{A}:X)_l.$ Now $X\Gamma Y \subseteq A \subseteq \overline{A} \Rightarrow Y \subseteq (\overline{A}:X)_r = A.$ Hence the theorem.

3.6. Theorem. Let R be a partial Γ-semiring. Then every 2(1)-prime partial ideal I of (R,Γ) contains a minimal 2(1)-prime partial ideal of (R,Γ).

Proof. Take $\tau = \{A \mid A$ is a 2-prime partial ideal of $(R,\Gamma) \ni A \subseteq I\}$. Clearly $I \in \tau$. Hence $(\tau, \subseteq)$ is a non-empty poset. Let $\{B_j \mid j \in J\}$ be a descending chain of elements in τ. Take $B := \bigcap_{j \in J} B_j$. Then B is clearly a partial ideal of (R,Γ) which contains I. Now let $x, y \in R \ni \overline{<x>}\Gamma\overline{<y>} \subseteq B$ and $x \notin B.$ Then $x \notin B_k$ for some $k \in J$. If $j \le k:$ Since $\overline{<x>}\Gamma\overline{<y>} \subseteq B_k$ and $x \notin B_k$, $y \in P_k$. Since $j \le k, P_k \subseteq P_j$ and hence $y \in B_j$ $\forall j \le k$. If $j \ge k:$ Then $B_j \subseteq B_k$. Since $x \notin B_k, x \notin B_j$. Now $\overline{<x>}\Gamma\overline{<y>} \subseteq B_j$ and $x \notin B_j$, $y \in B_j$ $\forall j \ge k$. Hence $y \in B_j$ for all $j \in J. \Rightarrow y \in \bigcap_{j \in J} B_j = B.$ Hence B is a 2-prime partial ideal of (R,Γ) which contains I. Thus B is a minimal of $\{B_j \mid j \in J\}$ in τ. By Zorn's lemma, τ has a minimal element. □

The set of all 2(1)-prime partial ideals of (R,Γ) is denoted by *2(1)-spec(R)*.

3.7. Theorem. Let R be a partial Γ-semiring with left/right unity. Then $\bigcap 2(1)-spec(R) = \{x \in R \mid$ there is an $m_2(m_1)$-system A of R with $x \in A$ implies $0 \in A\}$.

Proof. Take $T := \{x \in R \mid$ there is an m_2-system A of R with $x \in A$ implies $0 \in A\}$. Let $x \notin \bigcap 2-spec(R)$. Then $x \notin P$ for some 2-prime partial ideal P of (R,Γ). Since P is a 2-prime partial ideal of (R,Γ), $R \setminus P$ is an m_2-system of (R,Γ). Since $0 \in P, 0 \notin R \setminus P$. Thus $\exists$ an m_2-system $R \setminus P \ni x \in R \setminus P$ and $0 \notin R \setminus P. \Rightarrow x \notin T$. Therefore $T \subseteq \bigcap 2-spec(R)$. Let $x \notin T$. Then $\exists$ an m_2-system A of R $\ni x \in A$ and $0 \notin A$. Now <0> is a partial ideal of $(R,\Gamma) \ni \overline{<0>} \cap A = \phi$. By theorem 3.5., there is a 2-prime partial ideal P $\ni <0> \subseteq P$ and $\overline{P} \cap A = \phi$ and P is maximal with respect to this property. Since $x \in A, x \notin \overline{P}$. Since $P \subseteq \overline{P}, x \notin P$ for some 2-prime partial ideal P of $(R,\Gamma). \Rightarrow x \notin \bigcap 2-spec(R)$. Therefore $\bigcap 2-spec(R) \subseteq T$. Hence $T = \bigcap 2-spec(R)$.. □

3.8. Definition. Let K be a proper partial ideal of a partial Γ-semiring R with left/right unity. Then the *2(1)-prime radical* of K is defined as the smallest 2(1)-prime partial ideals which contains K and is denoted by $2(1)-\sqrt{K}$. i.e., $2(1)-\sqrt{K} = \bigcap\{P \in 2(1)-spec(R) \mid \overline{K} \subseteq P\}$.

3.9. Theorem. If K is a proper partial ideal of a partial Γ-semiring R then $2(1)-\sqrt{K} = \{x \in R \mid$ there is an $m_2(m_1)$-system A of R with $x \in A$ implies $A \cap \overline{K} \neq \phi\}$.

Proof. Take $T := \{x \in R \mid$ there is an m_2-system A of R with $x \in A$ implies $A \cap \overline{K} \neq \phi\}$. Let $x \notin T$. Then $\exists$ an m_2-system A of R $\ni x \in A$ and $A \cap \overline{K} = \phi$. By theorem 3.5., $\exists$ a 2-prime partial ideal P of R $\ni K \subseteq P$ and $A \cap \overline{P} = \phi$. Since $x \in A, x \notin \overline{P}. \Rightarrow x \notin 2-\sqrt{K}$. Therefore $2-\sqrt{K} \subseteq T$. Let $x \notin 2-\sqrt{K}$. Then $\exists$ a 2-prime partial ideal P of R $\ni K \subseteq P$ and $x \notin P. \Rightarrow x \in R \setminus P$ and $(R \setminus P) \cap \overline{K} = \phi. \Rightarrow \exists$ an m_2-system $R \setminus P$ of R $\ni x \in R \setminus P$ and $(R \setminus P) \cap \overline{K} = \phi. \Rightarrow x \notin T$. Therefore $T \subseteq 2-\sqrt{K}$. Hence the theorem.

IV. CONCLUSION

In this paper we introduced the notions of 2(1)-prime partial ideal and $m_2(m_1)$-system in a partial Γ-semiring and proved that P is a 2(1)-prime partial ideal of R if and only if R\P is an $m_2(m_1)$-system of (R,Γ). Also we obtained the characterizations of intersection of all 2(1)-prime partial ideals of *R* and 2(1)-prime radical of *K*.

REFERENCES

[1] Acharyulu, G.V.S.: A Study of Sum-Ordered Partial Semirings, Doctoral thesis, Andhra University , 1992.

[2] Acharyulu, G.V.S.: *Matrix Representable So-rings*, Semigroup Forum, Vol. 46, 31-47, 1993.

[3] Arbib, M.A., Manes, E.G.: Partially Additive Categories and Flow-diagram Semantics, Journal of Algebra, Vol. 62, 203-227, 1980.

[4] Nanda Kumar, P.: 1-(2-) Prime Ideals in Semirings, Kyungpook Mathematical Journal, Vol. 50, 2010, 117-122.

[5] Manes, E.G., and Benson, D.B.: The Inverse Semigroup of a Sum-Ordered Partial Semiring, Semigroup Forum, Vol. 31, 129-152, 1985.

[6] Murali Krishna Rao, M.: Γ-semirings-I, Southeast Asian Bulletin of Mathematics, Vol. 19(1), 49-54, 1995.

[7] Murali Krishna Rao, M.: Γ-semirings-II, Southeast Asian Bulletin of Mathematics, Vol. 21, 281-287, 1997.

[8] Siva Mala, M., Siva Prasad, K.: Partial Γ-Semirings, Southeast Asian Bulletin of Mathematics, Vol. 38, 873-885, 2014.

[9] Siva Mala, M., Siva Prasad, K.: (ϕ, ρ)-Representation of Γ-So-Rings, Iranian Journal of Mathematical Sciences and Informatics, Vol. 10(1), 103-119, 2015.

[10] Siva Mala, M., Siva Prasad, K.: Ideals of Sum-Ordered partial Γ-Semirings, Southeast Asian Bulletin of Mathematics, Vol. 40, 413-426, 2016.

[11] Siva Prasad, K., Siva Mala, M. & Srinivasa Rao, P.V.: Green's Relations in Partial Γ-Semirings, International Journal of Algebra and Statistics(IJAS), Vol. 2(2), 21-28, 2013.

[12] Siva Mala, M. & Siva Prasad, K.: Prime Ideals of Γ-So-rings, International Journal of Algebra and Statistics(IJAS), Vol. 3(1), 1-8, 2014.

[13] Siva Mala, M. & Siva Prasad, K.: Semiprime Ideals of Γ-So-rings, International Journal of Algebra and Statistics(IJAS), Vol. 3(1), 26-33, 2014.

[14] Streenstrup, M.E.: Sum-Ordered Partial Semirings, Doctoral thesis, Graduate school of the University of Massachusetts, Feb 1985.

