Invariant submanifolds of LCS-manifold admitting semi-symmetric metric connection

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Abstract: In this paper, we study invariant submanifolds of (LCS)-manifold admitting semi-symmetric metric connection. We prove that the submanifolds also carry the semi-symmetric metric connection. Further, we also consider recurrent, 2-recurrent and generalized 2-recurrent submanifolds of (LCS)-manifold admitting semi-symmetric metric connection and we investigate the conditions for all the above mentioned submanifolds to be totally geodesic.

Key Words: Invariant submanifold, (LCS)-manifold, recurrent, 2-recurrent, generalized 2-recurrent and totally geodesic.

I. INTRODUCTION

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with metric \(g\) and Let \(\nabla\) be the Levi-civita connection on \(M\). A linear connection \(\overline{\nabla}\) on \((M, g)\) is said to be Semi-symmetric [1] if the torsion tensor \(T\) of connection \(\overline{\nabla}\) satisfies

\[ T(X, Y) = W(X)Y - W(Y)X \]

where \(W\) is a 1-form on \(M\). Further let \(\rho\) be the vector field associated with it, that is \(W(X) = g(X, \rho)\) for any differentiable vector field \(X\) on \(M\). A Semi-symmetric linear connection \(\overline{\nabla}\) is called Semi-symmetric metric connection [2] if it satisfies \(\overline{\nabla}g = 0\). In 1924, Friedman and Schouten [1] introduced the idea of semi-symmetric linear connection on a differentiable manifold. Further in 1932, The idea of metric connection with torsion on a Riemannian manifold was introduced by H.A. Hayden [2]. Later, Yano [3] synthesized the notion of semi-symmetric connection and a metric connection with torsion on a Riemannian manifold in 1970. After the properties of semi-symmetric metric connection have been studied by many authors like K.S. Amur and S.S. Pujar [4], C.S. Bagewadi, D.G. Prakash and Venkatesha [5,6], A. Sharifuddin and S.I. Hussain [7], U.C. De and G. Pathak [8] etc. Recently, in 2003 A.A. Shaikh [9] introduced and studied Lorentzian concircular structure manifold (briefly (LCS)-manifold) which generalizes the notion of LP-sasakian manifolds, introduced by Matsumoto [10] in 1989. Then Shaikh and Baishya [11,12] have been studied on the applications of \((LCS)_n\) manifolds to the general theory of relativity and cosmology. It is to be noted that \((LCS)_n\) manifold remains invariant under a D-homothetic transformation [13]. On the other hand in 2008, A.A. Shaikh, T.Basu and S. Eyasmin [20] proved that existence of \(\phi\)-recurrent \((LCS)_1\) manifold which is neither locally symmetric nor locally \(\phi\)-symmetric by non trivial examples. Later, G.T. Shreenivasa, Venkatesha, C.S. Bagewadi [23] also studied \((LCS)_{2n+1}\)-manifolds in 2009. Further in 2011 Atceken and Hui have been studied on the submanifold of an \((LCS)_n\). The \((LCS)\) manifolds have been also studied by Atceken [14], Narain and Yadav [15], Prakash [16], Shaikh [17], Shaikh et al. [18,19], Shaikh and Bihm [21], shaikh and Hui [21], Sreenivasa et al. [22], Yadav et al. [23] and others. Recently, in 2016 Shaikh, Matsuyama and Hui have been studied on invariant submanifolds of \((LCS)_n\)-manifolds and it deals with the study of some basic properties of invariant submanifolds of \((LCS)_n\)-manifolds.

If \(\nabla\) denotes semi-symmetric metric connection on a contact metric manifold, then it is given by [13]

\[ \overline{\nabla}X = \nabla Y + \eta(Y)X - g(X,Y)\xi, \quad (1.1) \]

where \(\eta(Y) = g(Y, \xi)\).

The covariant differential of the \(p^{th}\) order, \(p \geq 1\), of a \((0, k)\)-tensor field \(T\), \(k \geq 1\), defined on a Riemannian manifold \((M, g)\) with the Levi-civita connection \(\nabla\), is denoted by \(\nabla^T\).

The tensor \(T\) is said to be recurrent if it satisfies

\[ (\nabla T)(X_1, ..., X_k; X) = (\mathcal{L}_X T)(X_1, ..., X_k; X) \]

On \(M\). where \(X, Y, X_1, Y_1, ..., X_k, Y_k \in \mathcal{TM}\). From (1.2) it follows that at a point \(x \in M\), if the tensor \(T\) is non zero, then there exist a unique 1-form \(\phi\) and a \((0, 2)\)-tensor \(\psi\), defined on a neighbourhood \(U\) of \(x\) such that

\[ \nabla^T = \mathcal{T} \phi, \quad \phi = d(log\|T\|). \]

Similarly, the tensor \(T\) is said to be \(2\)-recurrent if it satisfies

(1.3)
(\nabla^2 T)(X_1, \ldots, X_k; X, Y)T(Y_1, \ldots, Y_k) = \\
(\nabla^2 T)(Y_1, \ldots, Y_k; X, Y)T(X_1, \ldots, X_k), \quad (1.4)

On M. Where \( X, Y, X_1, Y_1, \ldots, X_k, Y_k \in TM \). Now from (1.4) it follows that at a point \( x \in M \), if the tensor \( T \) is non zero, then there exist a \((0, 2)\)-tensor \( \psi \), defined on a neighborhood \( U \) of \( x \) such that

\[
\nabla^2 T = T \otimes \psi, \quad (1.5)
\]

hold on \( U \), where \( \| T \| \) denotes the norm of \( T \) and \( \| T \|^2 = g(T, T) \). The tensor \( T \) is said to be generalized 2-recurrent if

\[
((\nabla^2 T)(X_1, \ldots, X_k; X, Y) - (\nabla^2 T)(Y_1, \ldots, Y_k; X, Y))T(Y_1, \ldots, Y_k) = \\
((\nabla^2 T)(Y_1, \ldots, Y_k; X, Y) - (\nabla^2 T)(X_1, \ldots, X_k; X, Y))T(X_1, \ldots, X_k), \quad (1.6)
\]

hold on \( M \), where \( \phi \) is a 1-form on \( M \). From (1.6) it follows that at a point \( x \in M \) if the tensor \( T \) is non zero, then there exists a \((0, 2)\)-tensor \( \psi \), defined on a neighborhood \( U \) of \( x \) such that

\[
\nabla^2 T = \nabla T \otimes \phi + T \otimes \psi, \quad (1.7)
\]

holds on \( U \).

## II. ISOMETRIC IMMERSSION

Let \( f: (M, g) \rightarrow (\tilde{M}, \tilde{g}) \) be an isometric immersion from an \( n \)-dimensional Riemannian manifold \((M, g)\) into \((n+d)\)-dimensional Riemannian manifold \((\tilde{M}, \tilde{g})\), \( n \geq 2 , \ d \geq 1 \). We denote by \( \nabla \) and \( \tilde{\nabla} \) as Levi-Civita connection of \( M^n \) and \( \tilde{M}^{n+d} \) respectively. Then the formulas of Gauss and Weingarten are given by

\[
\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad (2.1)
\]

\[
\tilde{\nabla}_X N = -A_N X + \tilde{\nabla}_X N, \quad (2.2)
\]

for any tangent vector fields \( X, Y \) and the normal vector field \( N \) on \( M \). Where \( \sigma \)-second fundamental form, \( A \)-the shape operator, \( \nabla^1 \)-is the normal connection.

If the second fundamental form \( \sigma \) is identically zero, then the manifold is said to be totally geodesic. The second fundamental form \( \sigma \) and \( A_N \) are related by

\[
\tilde{g}(\sigma(X, Y), N) = g(A_N X, Y), \quad (2.3)
\]

for tangent vector fields \( X, Y \).

The first and second covariant derivatives of the second fundamental form \( \sigma \) are given by

\[
(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla^1_X (\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \quad (2.4)
\]

\[
(\nabla^2 \sigma)(Z, W, X, Y) = (\tilde{\nabla}_X \nabla_Y \sigma)(Z, W), \quad (2.5)
\]

respectively, where \( \tilde{\nabla} \) is called the Vander Waerden-Bortolotti connection of \( M \) [7].

If

\[
\tilde{\nabla}_\sigma = 0
\]

then \( M \) is said to have parallel second fundamental form [7].

## III. LCS-MANIFOLD

In 2003 A. A. Shaikh the first author introduced the notion of Lorentzian concircular structure manifold (briefly, \( LCS \)-manifolds) and these are applied in General theory of relativity and cosmology.

An \( n \)-dimensional Lorentzian Manifold \( M \) is a smooth connected para-compact Housdorff manifold with a Lorentzian metric \( g \) that is, \( M \) admits a smooth symmetric tensor field \( g \) of type \((0, 2)\) such that for each point \( p \in M \), the tensor \( g_p: T_p M \times T_p M \rightarrow R \) is a non-degenerate inner product of signature \((- , +, +, +)\), where \( T_p M \) denotes the tangent vector space of \( M \) at \( p \) and \( R \) is the real number space. A non-zero vector \( v \in T_p M \) is said to be timelike if it satisfies \( g_p(v, v) < 0 \). The category to which a given vector fields is called its causal character.

In a Lorentzian manifold \((M, g)\), a vector field \( P \) defined by \( g(X, P) = A(X) \), for any \( X \) on \( M \), is said to be a concircular vector field if

\[
(\tilde{\nabla}_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}, \quad (3.1)
\]

for \( Y \) on \( M \), where \( \alpha \) is a non-zero scalar and \( \omega \) is a closed 1-form.

Let \( M \) be Lorentzian manifold admitting a unit timelike concircular vector field \( \xi \), called the structure vector field of the manifold. Then we have

\[
g(\xi, \xi) = -1, \quad (3.2)
\]

Since \( \xi \) is a unit concircular vector field, it follows that there exist a non-zero 1-form \( \eta \) such that

\[
g(X, \xi) = \eta(X), \quad (3.3)
\]

The following equation \( (\tilde{\nabla}_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \),

holds for all vector fields \( X, Y \) on \( M \) and \( \alpha \) is a non-zero scalar function satisfies

\[
\tilde{\nabla}_X \alpha = X(\alpha) = d\alpha(X) = \rho \eta(X), \quad (3.5)
\]

\( \rho \) being a certain scalar function given by

\[
\rho = -\xi(\alpha), \quad (3.6)
\]
If we put
\[ \nabla_X \xi = \alpha \varphi X, \]  
(3.7)

Then from (3.4) and (3.7) we have
\[ \varphi X = X + \eta(X) \xi, \]  
(3.8)

From which it follows that
\[ \varphi^2 = X + \eta(X) \xi, \]  
(3.9)

that is, \( \varphi \) is a symmetric (1,1) tensor field, called the structure tensor of the manifold. The n-dimensional Lorentzian manifold \( M \) together with the unit timelike concircular vector field \( \xi \), its associated 1-form \( \eta \), and an (1,1) tensor field \( \varphi \) is said to be a Lorentzian concircular structure manifold (briefly, \((LCS)_n\)-manifold). Especially, if \( a = 1 \), then we can obtain the LP-Sasakian structure of Matsumoto.

In \((LCS)_n\)-manifold, the following results hold
\[ \eta(\xi) = g(\xi, \xi) = -1, \phi^2 X = X + \eta(X) \xi, g(X, \xi) = \eta(X), \phi \xi = 0, \eta \circ \phi = 0, \]  
(3.10)

\[ g(\phi X, \phi Y) = g(X, Y) + \eta(X) \eta(Y), \]  
(3.11)

\[ \eta(R(X, Y)Z) = (\alpha^2 - \rho)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)), \]  
(3.12)

\[ S(X, \xi) = (\alpha^2 - \rho)(n - 1) \eta(X), \]  
(3.13)

for any vector fields \( X, Y, Z \) on \( M \) and \( \alpha^2 - \rho \neq 0 \), where \( R \)-curvature tensor and \( S \)-Ricci tensor of the manifold.

In an \((LCS)_n\)-manifold, we also have the following results
\[ (\nabla_X \eta)(Y) = (\nabla_Y \eta)(X), \]  
(3.14)

\[ d\eta(X, Y) = 0. \]  
(3.15)

We also mention that, in an \((LCS)_n\)-manifold the symmetric (1,1) tensor field \( \phi \) is idempotent and hence the eigen value of \( \phi \) is either 1 or 0.

IV. INARIANT SUBMANIFOLDS OF \((LCS)_n\)-MANIFOLD ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

If \( \tilde{M} \) is a \((LCS)_n\)-manifold with structure tensors \((\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{\varphi})\) then we know that its invariant submanifold \( M \) has the induced \((LCS)_n\)-structure \((\phi, \xi, \eta, \varphi)\).

A submanifold \( M \) of a \((LCS)_n\)-manifold \( \tilde{M} \) with a semi-symmetric metric connection is called an invariant submanifold of \( \tilde{M} \) with a semi-symmetric connection, if for each \( x \in M, \phi(T_x M) \subset T_x M \). The study of such submanifolds for the different type of contact manifolds have been carried out by authors of [4, 12, 26-], [29], [33, 37].

As a consequence, \( \xi \) becomes tangent to \( M \). For an invariant submanifold of a \((LCS)_n\)-manifold with a semi-symmetric metric connection, We have
\[ \sigma(X, \xi) = 0, \]  
(4.1)

for any vector \( X \) tangent to \( M \).

Let \( \tilde{M} \) be a \((LCS)_n\)-manifold admitting a semi-symmetric metric connection \( \tilde{\nabla} \cdot \)

Lemma 4.1. Let \( M \) be an invariant submanifold of contact metric manifold \( \tilde{M} \) which admits semi-symmetric connection \( \tilde{\nabla} \) and let \( \sigma \) and \( \tilde{\sigma} \) be the second fundamental form with respect to Levi-Civita connection and semi-symmetric metric connection then
(a) \( M \) admits semi-symmetric metric connection,
(b) the second fundamental form with respect to \( \tilde{\nabla} \) and \( \tilde{\nabla} \) are equal.

Proof. : we know that the contact metric structure \((\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{\varphi})\) on \( \tilde{M} \) induces \((\phi, \xi, \eta, \varphi)\) on invariant submanifold. By virtue of (1.1), we get
\[ \tilde{\nabla}_X Y = \nabla_X Y + \eta(Y) X - g(X, Y) \xi. \]  
(4.2)

By using (2.1) in (4.2)
\[ \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) + \eta(Y) X - g(X, Y) \xi. \]  
(4.3)

Now gauss formula (2.1) with respect to semi-symmetric metric connection is given by
\[ \tilde{\nabla}_X Y = \nabla_X Y + \tilde{\sigma}(X, Y). \]  
(4.4)

Equating (4.3) and (4.4), we get (1.1)
\[ \tilde{\sigma}(X, Y) = \sigma(X, Y). \]  
(4.5)

V. RECURRENT INARIANT SUBMANIFOLDS OF \((LCS)_n\)-MANIFOLD ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

We consider invariant submanifolds of a \((LCS)_n\)-manifold when \( \sigma \) is recurrent, 2-recurrent, generalized 2-recurrent and \( M \) has parallel third fundamental form with respect to semi-symmetric metric connection. We write the equations (2.4) and (2.5) with respect to semi-symmetric metric connection in the form
\[ \tilde{\nabla}_{X \sigma}(Y, Z) = \nabla_{\tilde{\nabla}X}(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \tilde{\nabla}_X Z), \]  
(5.1)

\[ \tilde{\nabla}_{X \tilde{\varphi}}(Z, W, X, Y) = (\tilde{\nabla}_X \tilde{\varphi})(\sigma)(Z, W) = \tilde{\nabla}_{\tilde{\varphi}}(\tilde{\nabla}_X \sigma)(Z, W) - \tilde{\nabla}_{\tilde{\varphi}}(\tilde{\nabla}_{\tilde{\varphi}}\sigma)(Z, W). \]  
(5.2)

Theorem 5.1. Let \( M \) be an invariant submanifold of \((LCS)_n\)-manifold admitting semi-symmetric metric connection. Then \( \sigma \) is recurrent with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.
Proof. Let \( \sigma \) be recurrent with respect to semi-symmetric metric connection. Then from (1.3) we get
\[
(\overline{\nabla}_X \sigma)(Y, Z) = \phi(X) \sigma(Y, Z),
\]
where \( \phi \) is a 1-form on \( M \). By using (5.1) and put \( Z = \xi \) in the above equation, we have
\[
\overline{\nabla}_X^\perp (\sigma(Y, Z) - \sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \xi)) = \phi(X) \sigma(Y, Z),
\]
which by virtue of (4.1) reduces to
\[
-\sigma(\overline{\nabla}_X Y, \xi) - \sigma(Y, \overline{\nabla}_X \xi) = 0. \tag{5.4}
\]
Using (1.1), (3.7), (3.10) and (4.1) in (5.4)
\[
-\alpha \sigma(Y, \phi X) + \sigma(Y, X) = 0. \tag{5.5}
\]
ow replace \( X \) by \( \phi X \) and by virtue of (3.10), (4.1) in (5.5), we get
\[
(1 - \alpha) \sigma(Y, X) = 0. \tag{5.6}
\]
We get \( \sigma(X, Y) = 0 \), if \( \alpha \neq 1 \) Therefore \( M \) is totally geodesic provided \( \alpha \neq 1 \)

**Theorem 5.2.** Let \( M \) be an invariant submanifold of a (LCS)-manifold \( \bar{M} \) admitting semi-symmetric metric connection. Then \( M \) has parallel third fundamental form with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.

**Proof.** Let \( M \) has parallel third fundamental form with respect to semi-symmetric metric connection. then we have
\[
(\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z, W) = 0
\]
Taking \( W = \xi \) and using (5.2) in the above equation, we have
\[
\overline{\nabla}_X \left( \left( \overline{\nabla}_Y \sigma(\xi, Z) \right) - \left( \overline{\nabla}_Y \sigma(Z, \xi) \right) - \left( \overline{\nabla}_X \sigma(Z, \xi) \right) \right) = 0. \tag{5.7}
\]
\[
0 = \overline{\nabla}_X^\perp \sigma(\overline{\nabla}_X Z, \xi) + \sigma(Z, \overline{\nabla}_X \xi) - \overline{\nabla}_Y^\perp \sigma(\overline{\nabla}_Y Z, \xi) + \sigma(\overline{\nabla}_Y Z, \overline{\nabla}_X \xi) + 2\sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \xi) + \sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \xi) + \sigma(Z, \overline{\nabla}_X \overline{\nabla}_Y \xi). \tag{5.8}
\]
In view of (1.1), (3.7), (3.10) and (4.1) in (5.8) gives
\[
0 = -2\overline{\nabla}_X^\perp \sigma(Z, \alpha \phi Y) + 2\overline{\nabla}_X^\perp \sigma(Z, Y) + \sigma(Z, \overline{\nabla}_X \alpha \phi Y) - \sigma(Z, \overline{\nabla}_X \alpha Y) \xi + \alpha \sigma(Z, \phi \overline{\nabla}_X Y) + \alpha \sigma(Y) \sigma(Z, \phi X) + \alpha \sigma(\overline{\nabla}_X Z, \phi Y) - \alpha \sigma(\overline{\nabla}_X \phi Y) + 2\alpha \sigma(\overline{\nabla}_X Z, \phi Y) - 2\sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \phi Y) - 2\eta(\overline{\nabla}_X Y) + 2\eta(\overline{\nabla}_X Z, \overline{\nabla}_Y \phi Y) - 2\eta(\overline{\nabla}_X \phi Y) \sigma(X, Y). \tag{5.9}
\]
Put \( Y = \xi \) and using (3.7), (3.10) and (4.1) in (5.9), we get
\[
0 = (\alpha^2 + 1)\sigma(Z, X) - 2\sigma(\phi, \phi X). \tag{5.10}
\]
Multiply by \( (\alpha^2 + 1) \) in (5.10), we get
\[
0 = (\alpha^2 + 1)^2 \sigma(Z, X) - 2\alpha(\alpha^2 + 1)\sigma(Z, \phi X). \tag{5.11}
\]
Put \( X = \phi X \) in (5.11), we get
\[
0 = (\alpha^2 + 1)\sigma(Z, \phi X) - 2\alpha(\alpha^2 + 1)\sigma(Z, \phi X). \tag{5.12}
\]
Multiply by \( 2\alpha \) in (5.12), we get
\[
0 = 2\alpha(\alpha^2 + 1)\sigma(Z, \phi X) - 4\alpha^2\sigma(Z, X). \tag{5.13}
\]
Adding equation (5.11) and (5.13), we get \( \alpha = \pm 1 \) and \( \sigma(X, Z) = 0 \). Thus \( M \) is totally geodesic. The converse statement is trivial. This proves the theorem

**Corollary 5.1.** Let \( M \) be an invariant submanifold of a (LCS)-manifold \( \bar{M} \) admitting semi-symmetric metric connection. Then \( \sigma \) is 2-recurrent with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.

**Proof.** Let \( \sigma \) be 2-recurrent with respect to semi-symmetric metric connection. From (1.5), we have
\[
(\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z, \xi) = 0
\]
Taking \( W = \xi \) and using (5.2) in the above equation, we have
\[
\overline{\nabla}_X \left( \left( \overline{\nabla}_Y \sigma(Z, \xi) \right) - \left( \overline{\nabla}_Y \sigma(Z, \xi) \right) - \left( \overline{\nabla}_X \sigma(Z, \xi) \right) \right) = 0. \tag{5.14}
\]
In view of (4.1) and (5.1) we write (5.14) in the form
\[
0 = \overline{\nabla}_X \left( \sigma(\overline{\nabla}_Y Z, \xi) + \sigma(\overline{\nabla}_Y Z, \xi) - \sigma(\overline{\nabla}_X Z, \xi) \right) + \sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \xi) + 2\sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \xi) + \sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \xi) + \sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \xi).
\]
(5.15)
Using (1.1), (3.7), (3.10) and (4.1) in (5.15), we get
\[
0 = -2\overline{\nabla}_X^\perp \sigma(Z, \alpha \phi Y) + 2\overline{\nabla}_X^\perp \sigma(Z, Y) + \sigma(Z, \overline{\nabla}_Y \alpha \phi Y) - \sigma(Z, \overline{\nabla}_Y \alpha Y) \xi + \alpha \sigma(Z, \phi \overline{\nabla}_X Y) + \alpha \sigma(Y) \sigma(Z, \phi X) + \alpha \sigma(\overline{\nabla}_X Z, \phi Y) - \alpha \sigma(\overline{\nabla}_X \phi Y) + 2\alpha \sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \phi Y) - 2\sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \phi Y) - 2\eta(\overline{\nabla}_X Y) + 2\eta(\overline{\nabla}_X Z, \overline{\nabla}_Y \phi Y) - 2\eta(\overline{\nabla}_X \phi Y) \sigma(X, Y).
\]
(5.16)
Put \( Y = \xi \) and using (3.7), (3.10) and (4.1) in (5.16), we get
\[
0 = (\alpha^2 + 1)\sigma(Z, X) - 2\alpha(\alpha^2 + 1)\sigma(Z, \phi X).
\]
Multiply by \( (\alpha^2 + 1) \) in (5.17), we get
\[
0 = (\alpha^2 + 1)^2 \sigma(Z, X) - 2\alpha(\alpha^2 + 1)\sigma(Z, \phi X).
\]
(5.18)
Put \( X = \phi X \) in (5.17), we get
\[
0 = (\alpha^2 + 1)\sigma(Z, \phi X) - 2\alpha(\alpha^2 + 1)\sigma(Z, \phi X).
\]
(5.19)
Multiply by \( 2\alpha \) in (5.19), we get
\[
0 = 2\alpha(\alpha^2 + 1)\sigma(Z, \phi X) - 4\alpha^2\sigma(Z, X).
\]
(5.20)
Adding equation (5.18) and (5.20), we get $\alpha \neq \pm 1$ and $\sigma(X, Z) = 0$. Thus $M$ is totally geodesic. The converse statement is trivial. This proves the theorem.

**Theorem 5.3.** Let $M$ be an invariant submanifold of a (LCS)-manifold $\tilde{M}$ admitting semi-symmetric metric connection. Then $\sigma$ is generalised 2-recurrent with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.

**Proof.** Let $\sigma$ be a generalised 2-recurrent with respect to semi-symmetric metric connection. From (1.7), we have

$$\tilde{\nabla}_\psi \tilde{\nabla}_\phi \sigma(Z, W) = \psi(X, Y) \sigma(Z, W) + \phi(X)(\tilde{\nabla}_\psi \sigma)(Z, W).$$

(5.21)

Where $\psi$ and $\phi$ are 2-recurrent and 1-form respectively. Taking $W = \xi$ in (5.21) and using (4.1), we get

$$\tilde{\nabla}_\psi \tilde{\nabla}_\phi \sigma(Z, \xi) = \phi(X)(\tilde{\nabla}_\psi \sigma)(Z, \xi).$$

Using (4.1), (5.1) and (5.2) in above equation, we get

$$\tilde{\nabla}_\psi \tilde{\nabla}_\phi \sigma(Z, \xi) - \tilde{\nabla}_\phi \sigma(\tilde{\nabla}_\psi \sigma)(Z, \xi) - \tilde{\nabla}_\psi \sigma(\tilde{\nabla}_\phi \sigma)(Z, \xi) = \phi(X)\sigma(\tilde{\nabla}_\psi Z, \xi) + \sigma(Z, \tilde{\nabla}_\phi \xi).$$

(5.22)

By using (4.1) and the virtue of (5.1), we write (5.22) in the form

$$\tilde{\nabla}_\psi \sigma(\tilde{\nabla}_\phi Z, \xi) + \sigma(\tilde{\nabla}_\phi \xi) = \tilde{\nabla}_\phi \sigma(\tilde{\nabla}_\psi Z, \xi) + 2\sigma(\tilde{\nabla}_\psi Z, \tilde{\nabla}_\phi \xi) - \tilde{\nabla}_\phi \sigma(\tilde{\nabla}_\psi Z, \tilde{\nabla}_\phi \xi) + \sigma(Z, \tilde{\nabla}_\psi Z, \tilde{\nabla}_\phi \xi) + \sigma(\tilde{\nabla}_\psi Z, \tilde{\nabla}_\phi \xi) - \sigma(Z, \tilde{\nabla}_\psi \tilde{\nabla}_\phi \xi) = \phi(X)\sigma(\tilde{\nabla}_\psi Z, \xi) + \sigma(Z, \tilde{\nabla}_\phi \xi).$$

(5.23)

In view of (1.1), (3.7), (3.10) and (4.1) in (5.23), we get

$$-2\tilde{\nabla}_\psi \sigma(\tilde{\nabla}_\phi Z, \xi) + 2\tilde{\nabla}_\phi \sigma(\tilde{\nabla}_\psi Z, \xi) + \sigma(\tilde{\nabla}_\phi \xi) - \sigma(Z, \tilde{\nabla}_\phi \xi) + \sigma(\tilde{\nabla}_\psi Z, \tilde{\nabla}_\phi \xi) + \sigma(Z, \tilde{\nabla}_\psi \tilde{\nabla}_\phi \xi) = \phi(X)\sigma(\tilde{\nabla}_\psi Z, \xi) + \sigma(Z, \tilde{\nabla}_\phi \xi).$$

(5.24)

Put $Y = \xi$ and using (3.7), (3.10) and (4.1) in (5.24), we get

$$0 = (\alpha^2 + 1)\sigma(X, Z) - 2\sigma(\tilde{\nabla}_\phi Z, \tilde{\nabla}_\phi X).$$

(5.25)

Multiply by $(\alpha^2 + 1)$ in (5.25), we get

$$0 = (\alpha^2 + 1)^2\sigma(X, Z) - 2\alpha(\alpha^2 + 1)\sigma(Z, \tilde{\nabla}_\phi X).$$

(5.26)

Put $X = \tilde{\nabla}_\phi X$ in (5.25), we get

$$0 = (\alpha^2 + 1)(\sigma(Z, X) - 2\sigma(\tilde{\nabla}_\phi Z, \tilde{\nabla}_\phi X)).$$

(5.27)

Multiply by $2\alpha$ in (5.27), we get

$$0 = 2\alpha(\alpha^2 + 1)\sigma(Z, \tilde{\nabla}_\phi X) - 4\alpha^2\sigma(Z, X).$$

(5.28)

Adding equation (5.26) and (5.28), we get $\alpha \neq \pm 1$ and $\sigma(X, Z) = 0$. Thus $M$ is totally geodesic. The converse statement is trivial. This proves the theorem.

**VI. CONCLUSION**

From the Theorems (5.1), (5.2) and (5.3) and from the Corollary (5.1), we conclude that:

If $M$ is an invariant submanifolds of LCS manifold admitting semi-symmetric metric connection then the following conditions are equivalent:

i. The submanifold $M$ is totally geodesic with respect to the Levi-Civita connection;

ii. Second fundamental form $\sigma$ is recurrent with respect to semi-symmetric metric connection;

iii. The submanifold $M$ has parallel third fundamental form with respect to semi-symmetric metric connection;

iv. Second fundamental form $\sigma$ is 2-recurrent with respect to semi-symmetric metric connection;

v. Second fundamental form $\sigma$ is generalised 2-recurrent with respect to semi-symmetric metric connection.

**REFERENCES**


